Einstein's Light Box and the Uncertainty Principle

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Abstract: Heisenberg's uncertainty principle is explained using the ideas of a wave packet and the true resolution to the riddle of Einstein's famous light box experiment is revealed.

It is ironic that the man who took the first steps along the road to quantum theory, namely Albert Einstein with his explanation of the photoelectric effect, was increasingly dismayed at the path which the theorists under the powerful influence of Niels Bohr were taking it. Throughout the 1920's, Einstein bombarded Bohr with thought experiments designed to show that his theory, and in particular the uncertainty principle discovered by Heisenberg, was fatally flawed but in each case Bohr was able to show that the measurement that Einstein proposed was not, in fact, possible and the uncertainty principle always held. Only once did Einstein cause Bohr any real trouble and that was during the sixth Solvay conference in Brussels in 1930 when Einstein presented Bohr with the following problem: imagine a box full of light; weigh the box; open a shutter for a short time ΔT; if a photon is seen to emerge, weigh the box again; the difference between the two weights will enable you to determine the energy of the photon (using the relation $E = mc^2$) with arbitrary precision; this contradicts the uncertainty principle which asserts that the product of the uncertainty in energy and the uncertainty in the time cannot be less than $h/2\pi$.

Bohr lay awake all night trying to come up with an answer to this riddle but he had his answer before breakfast. The flaw in the argument, he said, was that weighing the box had to take place in a gravitational field but the inevitable uncertainty about the position of the box in the field meant that, according to Einstein's own theory of General Relativity, there must be an equivalent uncertainty in the time at which the measurement was made and so the uncertainty principle was upheld. Apparently, Einstein accepted his solution.

This incident has often been portrayed by popular historians as the final clash between two intellectual giants which resulted in the complete triumph of Bohr's interpretation of quantum theory and Einstein's total defeat. Now while Bohr's supporters undoubtedly hyped up the significance of the event, the truth is that Bohr's supposed solution to the riddle is at best unnecessary and possibly completely wrong; and both men failed to grasp the true significance of the uncertainty principle. To understand why we need to understand the fundamental reason why Heisenberg's uncertainty principle exists in the first place.

The Uncertainty Principle explained

The uncertainty principle is a direct consequence of the wave/particle duality which asserts that a single photon (or, indeed, any quantum particle) can, under certain circumstances, be regarded as a short burst of waves called a wave packet. Consider the two wave packets shown in *fig.*1 and *fig.*2 which have roughly the same wavelength (and therefore energy and momentum) but different duration:

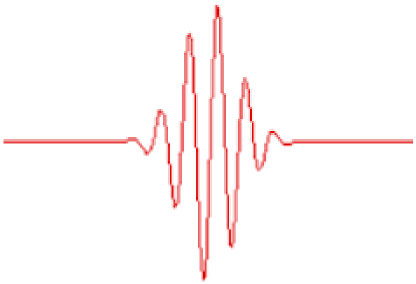


fig. 1: Short wave packet

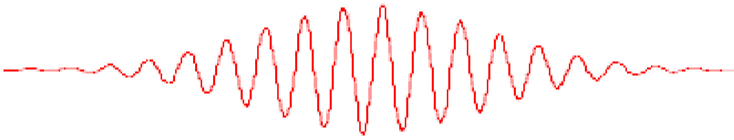


fig. 2: Long wave packet

Now lets try measuring their respective position and wavelength. The upper one contains about 5 (± ½ a wavelength) wavelengths in about 30 mm[1] so its wavelength is approximately 6mm. The *uncertainty* in the wavelength is mainly due to the uncertainty in counting the number of wavelengths in the packet and this is 10% so our measurement of wavelength is 6.0 ± 0.6 mm. Carrying out the same analysis on the longer wave packet we find that it contains 15 ± ½ wavelengths in 90 mm. The measured wavelength is therefore also 6 mm but its uncertainty is less because we have counted more wavelengths. The uncertainty is about ± 0.2 mm. (A wave packet like this does not actually have a single wavelength; it contains a range of wavelengths clustered around a single peak so in the latter case, what we are really saying is that the wave packet contains a range of wavelengths ranging from about 5.8 mm to 6.2 mm.)

Now lets consider the uncertainty in *position*. Due to the difficulty of determining where the packet begins and ends, the uncertainty in position is probably of the order of one third its length, namely 10 mm for the short one and 30 mm for the long one.

The next step is to multiply the uncertainty in wavelength ($\Delta\lambda$) by the uncertainty in position (Δx). For the short one the answer is $0.6 \times 10 = 6\ \text{mm}^2$ while for the long one it is $0.2 \times 30 = 6\ \text{mm}^2$. Not surprisingly, the answers are exactly the same because the longer the wave packet, the more accurately you can measure its wavelength – but the less accurately you can determine its position in exact proportion.

Where does the magic figure of 6 mm^2 come from? Well, suppose that the wave packet contains n waves which we can count to an accuracy of ± ½ a wavelength. It is easy to show that the uncertainty in our measurement of wavelength (λ) will be $\lambda/2n$. As for the uncertainty in position that is something of the order of $n\lambda/3$ and the product of these two uncertainties is $\lambda^2/6$. The n's cancel out. A more sophisticated argument replaces the figure 6 with 2π.

i.e. $$\Delta x \Delta\lambda = \lambda^2/2\pi \qquad (1)$$

Now since the momentum $$p = h/\lambda \qquad (2)$$

by differentiating (2) $$\Delta p = -\frac{h}{\lambda^2}\Delta\lambda \qquad (3)$$

Now from (1) $$\Delta x = \frac{\lambda^2}{2\pi\Delta\lambda} \qquad (4)$$

so $$\Delta p \Delta x = (\text{-})\frac{h}{2\pi} \qquad (5)$$

Alternatively we can use the other quantum relation

$$E = h\nu = hc/\lambda \qquad (6)$$

hence $$\Delta E = -\frac{hc}{\lambda^2}\Delta\lambda \qquad (7)$$

1 Note that owing to the reduction in scale when this article is printed, the actual measurements made on the printed page will differ from those I have quoted but the conclusions will be the same.

Since photons move at the speed of light c, $\Delta T = \Delta x / c$ so

$$\Delta T = \frac{\lambda^2}{2\pi c \Delta\lambda} \tag{8}$$

$$\Delta E \Delta T = (\text{-}) \frac{h}{2\pi} \tag{9}$$

Equations (5) and (9) represent Heisenberg's famous uncertainty principle and, as we have seen, they are a direct consequence of the wave nature of a quantum particle. Nothing else.

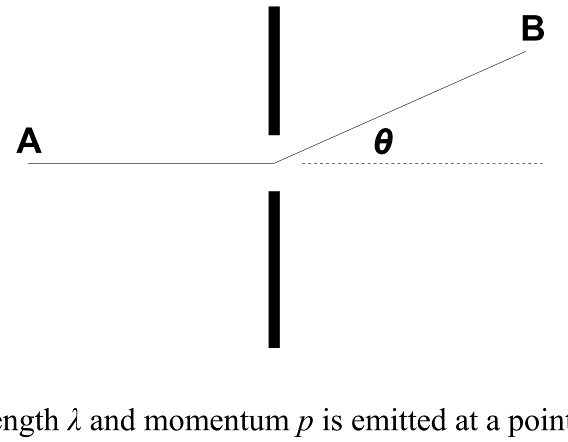
Einstein's Light Box Experiment

If it really is true that Heisenberg's uncertainty principle is just a consequence of the wave/particle duality, then it *must* be possible to explain why Einstein's experiment is flawed using just this concept. Gravity should not be an issue, nor should relativistic time dilation. In the first place, it would be possible to carry out the experiment in a zero gravity environment (using some kind of inertial measurement to measure the mass of the box); and second, there is no reason why we should not imagine the experiment taking place in a non-relativistic context (e.g. by using neutrons instead of photons). What is more, if it was truly that case that the Uncertainty Principle relied on General Relativity for its consistency, then this would point to a fundamental connection between the two theories which subsequent efforts in the search for a viable quantum theory of gravity have notably failed to find.

So if the answer to the riddle is not to be found by invoking Relativity, where exactly is the flaw in Einstein's argument?

Well, the uncertainty in the time at which the photon is emitted is definitely ΔT. There can be no argument about that. At this point Einstein confused the issue by measuring the energy of the photon by weighing (or measuring the mass of) the box before and after the experiment and using his relation $E = mc^2$. Einstein assumed that it would be possible to measure the masses with arbitrary position and it was this aspect of his argument that Bohr attacked. But this is not the point. Einstein is perfectly correct in saying that the mass of the box can be measured with arbitrary precision and he is also correct is saying that the energy of the emitted photon can be measured with arbitrary precision. In fact there is nothing to stop us chasing after the photon and measuring its energy by more conventional means with arbitrary precision. Where both Einstein and Bohr went wrong was to infer that these two 'measurements' are incompatible with the Uncertainty Principle.

In order to clarify the issue, let us consider a situation with which we are much more familiar – the diffraction of a photon at a single slit.



A single photon of wavelength λ and momentum p is emitted at a point A, passes through a single slit of width Δx and is detected at a point B having been diffracted through an angle θ.

Elementary wave theory tells us that then central maximum of the diffraction pattern lies within t region ± φ where $\sin\phi = \frac{\lambda}{\Delta x}$.

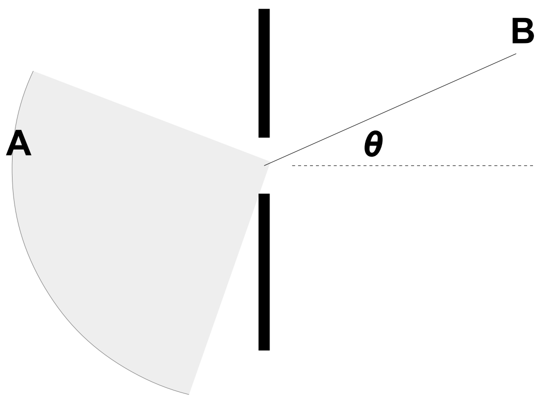
Another way of interpreting this situation is to say that, since the photon is constrained in its X position to an accuracy of Δx as it passes through the slit, the Uncertainty Principle insists that uncertainty in the momentum in the X direction, Δp, cannot be less than $\frac{h}{2\pi\Delta x}$. Since the momentum of the photon is $p = h/\lambda$ the uncertainty in the direction in which the photon is travelling cannot be less than $\frac{\Delta p}{p} = \frac{h}{2\pi\Delta x\, p} = \frac{\lambda}{2\pi\Delta x}$ and that therefore the angle through which the photon is diffracted is likely to be of the order of $\frac{\lambda}{\pi\Delta x}$

Note that I am not suggesting the the Uncertainty Principle is *responsible* for the diffraction of light. All I am saying is that the diffraction of light is *consistent* with the UP.

Now we can measure θ to an arbitrary degree of accuracy and hence we can 'measure' the momentum of the photon in the X direction to an arbitrary degree of accuracy as well; the width of the slit can be made arbitrarily small too, so it would appear that we have measured both the position and the momentum of the photon as it leaves the slit to an arbitrary degree of accuracy in apparent violation of the UP. But clearly this is not the case because, as we have just shown, diffraction is perfectly consistent with the UP.

In exactly the same way, Einstein argued that you could 'measure' the time of exit of the photon from the box and its energy as precisely as you wish but both Einstein and Bohr failed to realise that these statements do not contradict the UP any more than does the phenomenon of diffraction.

It is worth pursuing this a little further to discover what statements *would* contradict the UP. Let us return to the single slit experiment and ask ourselves, what exactly can we deduce from the fact that a photon undeniably passed through the slit and landed at B? Can we deduce that it started from A? Absolutely not. All we can deduce it that the photon probably originated somewhere within the shaded arc shown in the following diagram.



In other words, while measuring θ tells us the momentum of the photon in the X direction *after* it has passed through the slit, it tells us a lot less about the momentum of the photon *before* it passed through the slit. *This* is the measurement which would violate the UP if only we could carry it out.

Similarly, the fact that we can measure the energy of Einstein's photon after it has emerged from the box with arbitrary accuracy is irrelevant. What happens when the photon passes through the shutter in time Δt is that it undergoes a spontaneous change in its energy ΔE where $\Delta t\, \Delta E < h/2\pi$, a phenomenon which could be called temporal diffraction.

This raises the interesting question of where this this energy comes from. Unlike quantum tunnelling, this energy is not 'paid back' but is real and must be accounted for. Again, a consideration of the single slit experiment will help. When the photon is diffracted through an angle θ it acquires momentum in the X direction which it did not have before. Where does this momentum come from? Well, the only thing that it interacted with on its journey was the slit so it follows that the slit must acquire an equal and opposite momentum. In the same way, if a photon acquires extra energy in passing through a shutter, the shutter must lose it – but in what way I am not entirely sure.

This analysis suggest an experiment which could in principle actually be performed. An ultra-stable laser whose frequency is known with great accuracy is directed at a Kerr cell shutter which is opened for a very short time. The energy of the photons passing through the shutter is measured very accurately. The UP dictates that the emitted photons will show a range of energies and, no doubt, a rigorous analysis of the situation using standard quantum theory will determine the expected range in detail. The problem with this experiment is that is is probably well beyond the limits of our current technology. In order to get a significant variation in photon energies, the shutter speed must be of the order of the period of the photon stream. For visible light this is around 10^{-15} s. The fastest shutter speed which a Kerr cell is capable of is around 10^{-9} s

In fact experiments have been carried out in which *matter waves* have exhibited diffraction effects when confined to certain temporal constraints. Wikipedia lists the following reference:

Szriftgiser, A.; Guéry-Odelin, D.; Arndt, M.; Dalibard, J. (1996). "Atomic Wave Diffraction and Interference Using Temporal Slits". *Physical Review Letters*. **77** (1): 4–7. Bibcode:1996PhRvL..77....4S. doi:10.1103/PhysRevLett.77.4. PMID 10061757

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