The Physics of Flight (1) - Fixed and Rotating wings

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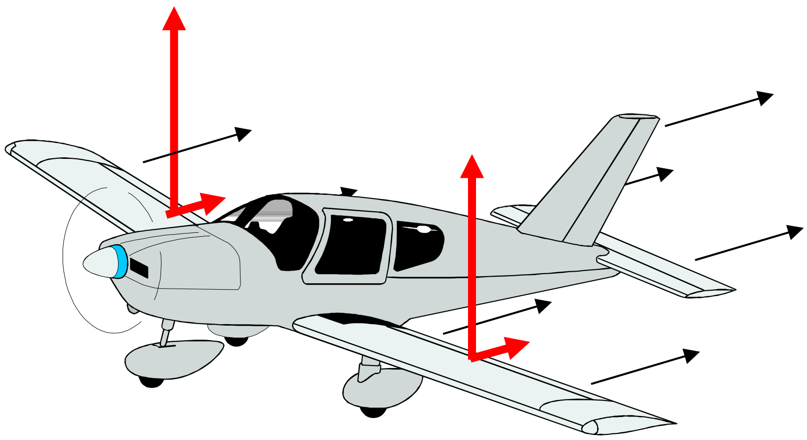
Abstract

Almost all elementary textbook explanations of the theory of flight rely heavily on Bernoulli's principle and the fact that air travels faster over a wing than below it. In recent years the inadequacies and indeed, fallacies in this explanation have been exposed (see Holger Babinsky's excellent article in Physics Education, November 2003[1]) and it is now appreciated that it is possible to provide a much simpler explanation in terms of Newton's laws. In this article it is shown how a couple of plausible assumptions are all that is needed to calculate a lot of interesting facts about wings.

Some aerodynamic definitions

When an aeroplane flies through air, the forces acting on it can be split into three different components – the lift and two sorts of drag; induced drag and parasitic drag.

Fig. 1



In the above diagram,. The force of lift is indicated by the vertical red arrows, the induced drag by the horizontal red arrows and the parasitic drag by the black arrows. As is suggested by the diagram, parasitic drag acts on the whole aircraft while induced drag only acts on the wing and is closely connected with the force of lift. We shall deduce the exact relation in due course. Let's take a closer look at the three forces.

Lift

Conventional aerodynamic theory tells us that the lift on a wing of area S_W moving with velocity v through a medium of density ρ is given by the equation

$$F_L = \tfrac{1}{2}\, C_L\, S_W\, \rho\, v^2 \qquad (1)$$

where C_L is a constant called the coefficient of lift. This constant is determined crucially by the shape of the wing and its angle of attack and for a well-designed wing it can reach a maximum value of about 1.5. The formula embodies the experimentally well verified fact that, to a large degree the lift on a wing is proportional to the square of the velocity of the air flowing over it.

In level flight (or at takeoff), the lift on the wing is equal to the weight (Mg) of the aircraft so we can work out the required speed of the aircraft from the equation

$$v = \sqrt{\frac{2\,Mg}{C_L S_W \rho}} \tag{2}$$

In order to work out the takeoff speed, all you have to do is set the coefficient of lift to its maximum value (which is, in practice about 1.5). Let us try putting in some figures. A small radio/controlled model aircraft ($M = 2$ kg, $S_W = 0.15$ m^2) has a takeoff speed of about 12 ms^{-1}. A Cessna 172 (M = 1100 kg, $S_W = 16.5$ m^2) takes off at 26 ms^{-1} and while a Boeing 747 ($M = 390$ tonnes, $S_W = 541$ m^2) needs a speed of 86 ms^{-1}.

You can also use the formula backwards

ie

$$C_L = \frac{2\,Mg}{S_W \rho\, v^2} \tag{3}$$

to calculate the required coefficient of lift for any given cruising speed. eg in order to cruise at 500 mph (222 ms^{-1}) the pilot of a Boeing 747 must reduce its coefficient of lift (by altering the angle of attack of the wings) to around 0.4. (This figure takes into account the reduced density of air at 30,000 feet)

Induced Drag

Whenever a wing produces lift, it deflects air downwards. The kinetic energy of this downward moving air is wasted and appears as a drag force on the wing which must be overcome by the plane's engines. Like lift, this force depends on the density of the air and the square of the velocity, so we can write an equation very similar to equation (1) but using a new coefficient, the coefficient of induced drag C_{ID}.

$$F_{ID} = ½\, C_{ID}\, S_W\, \rho\, v^2 \tag{4}$$

Since Power = force × velocity,

$$P_{ID} = F_{ID}\, v \tag{5}$$

$$P_{ID} = ½\, C_{ID}\, S_W\, \rho\, v^3 \tag{6}$$

Parasitic Drag

Parasitic drag is caused by a number of different factors but the most significant (at least for small aeroplanes and birds flying at relatively low speeds) is the force of the air hitting the frontal area of the plane. This force is also proportional to density and the square of the velocity so we can define a third coefficient C_{PD} (the coefficient of parasitic drag) in a similar way to the others – the big difference being that the area in the equation (S_F) is not the area of the wing, but the frontal area of the plane:

$$F_{PD} = ½\, C_{PD}\, S_F\, \rho\, v^2 \tag{7}$$

As with induced drag, the power required to overcome this force is given by the equation:

$$P_{PD} = ½\, C_{PD}\, S_F\, \rho\, v^3 \tag{8}$$

Obviously a well designed aeroplane will be as streamlined as possible to make C_{PD} as small as possible but it cannot be ignored. At low speeds and high angles of attack (eg on takeoff) induced drag is the dominant factor but when cruising, parasitic drag becomes more important.

The total power required to fly a plane is the sum of the above two expressions (6) and (8).

So far we have only been playing around with some definitions. Now for the Physics.

Newton's laws

As the wing moves forward through the air, it deflects some of the air downwards – ie it gives it a vertical momentum which it did not have before. By Newton's laws, the lift on the wing is exactly equal to the rate of change of momentum given to the air, and the power required to drive the wing forward P_{ID} will be equal to the rate at which the air is given kinetic energy.

Suppose that every second, a mass m_{air} is given a vertical velocity v_{air} downwards. The lift on the wing is given by

$$F_L = m_{air} \ v_{air} \tag{9}$$

and the rate of change of kinetic energy (equal to the power required to drive the wing through the air is

$$P_{ID} = ½ \, m_{air} \ v_{air}^2 \tag{10}$$

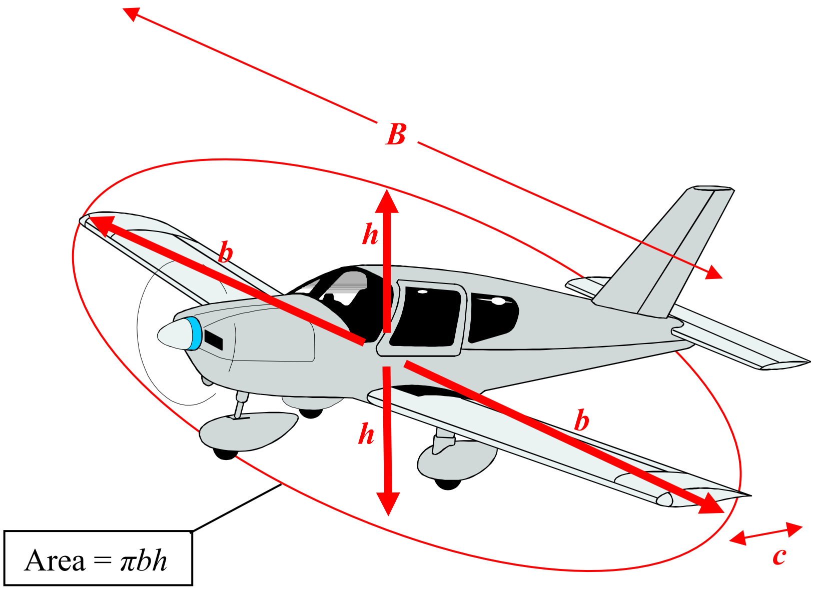
Eliminating v_{air} from these equations gives us

$$P_{ID} = \frac{F_L^{\ 2}}{2\, m_{air}} \tag{11}$$

But when we try to write down an expression for m_{air} , we run against a difficulty. How much air is affected when the wing travels along?

The fundamental assumption of the theory used in this article is that when a wing moves through still air, it affects an elliptical prism of air whose cross sectional area is related in some way to the wingspan of the aircraft *B* (or the semi-span *b)* and the mean wing chord *c* (the mean distance from the leading edge to the trailing edge). The relevant dimensions are illustrated in figure 2.

Fig.2



It seems reasonable to suppose that the height of the ellipse *h* will be a function of both *b* and *c*. But what function? It can't be the mean of *b* and *c* because *h* must be zero when *either b or c* is zero. Nor can it increase without limit as *b* increases. The correct answer turns out to be the *reciprocal* mean of *b* and *c*. (i.e. the reciprocal of the mean of the reciprocals of *b* and *c* - more easily calculated from the formula 2*bc* / (*b* + *c*)) Unlike the more familiar root mean square of two numbers, the reciprocal mean has the effect of favouring the *smaller* of the two values. Moreover, as one of the values tends to infinity, the value of the mean is limited to twice the other value.

Now every second, the aircraft moves forwards by a distance v so the mass of air affected every second is given by:

$$m_{\text{air}} = \pi b h \rho v = 2\pi b^2 c \rho v / (b + c) \tag{12}$$

The mean wing chord c is not always easy to measure, particularly in the case of birds, so we shall find it more convenient to restructure this relation using the total wing area $S_W = Bc = 2bc$ and the *aspect ratio* of the wing $A = B/c = 2b/c$.

This leads us to a fundamental relation determining the power (excluding parasitic power) needed to fly an aeroplane at any given speed:

$$P_{ID} = \frac{M^2 g^2 (1 + 2/A)}{2\pi \, S_W \rho \, v} \tag{13}$$

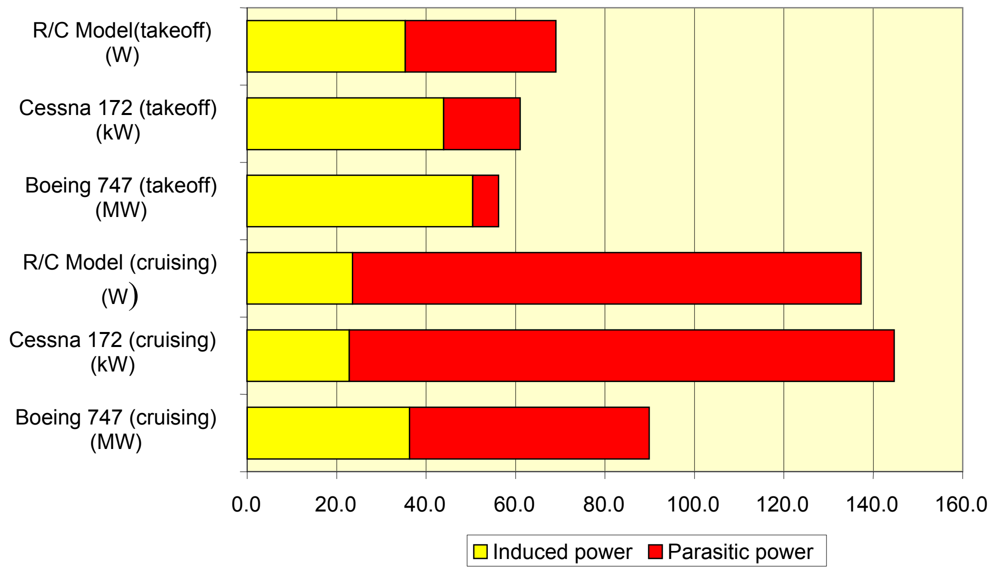
This is a truly remarkable result and requires a lot of explaining. Firstly it is not particularly surprising that the heavier the plane the greater the power needed – but we need to understand clearly what is being held constant. In order to be able to fly a heavier aeroplane with the same wing and at the same speed, we are going to have to increase the coefficient of lift by increasing the angle of attack. This will increase the drag and hence the required power.

If we fix the weight of the plane and its cruising speed we see that increasing the aspect ratio *decreases* the power needed to fly it. This is because a long wing will affect more air as it moves along but give it a smaller vertical velocity. The rate of change of momentum ($m_{air} \; v_{air}$) will remain the same but the rate of production of kinetic energy (½ $m_{air} \; v_{air}^2$) will be less because the velocity term is squared. This is why the Voyager (The plane that Rutan and Yeager flew round the world on one tank of fuel) had such incredibly long, thin wings.

Ii is also fairly obvious that increasing the wing area S allows the pilot to use a smaller angle of attack and the required power reduces, but the most surprising aspect of the equation is the presence of the speed v on the *bottom* of the equation. How can *increasing* the speed *decrease* the power required? The answer is that, like increasing the area, by flying faster, the angle of attack can be reduced, thus reducing the induced drag. Of course, the parasitic drag increases greatly with speed but there will be a certain speed for every aircraft or bird at which the total power required is a minimum

Fig. 3 compares the induced power with the parasitic power needed. On takeoff, induced drag dominates but at cruising speed, induced drag decreases as the angle of attack is reduced while parasitic drag increases dramatically. (In order to put the three aircraft on the same chart, power for the model is in W, the Cessna is in kW and the 747 is in MW)

Fig. 3



The relation between lift and induced drag

Now lets see what we can discover about the coefficients of lift and drag. First we equate expressions (6) and (13)

$$\tfrac{1}{2} C_{ID} S_W \rho v^3 = \frac{M^2 g^2 (1 + 2/A)}{2\pi\, S_W \rho\, v} \tag{14}$$

and then use equation (2) to eliminate v. All sorts of things cancel (try it!) and we are left with the following expression:

$$C_{ID} = \frac{C_L^{\,2} (1 + 2/A)}{4\pi} \tag{15}$$

Once again, this relation justifies close scrutiny. As before, we see that, up to a point, the greater the aspect ratio A the smaller the drag. This is why gliders have wings with very high aspect ratios.

Another consequence of great importance is that C_{ID} is proportional to the *square* of the coefficient of lift. At takeoff, the coefficient of lift is required to be high in order to minimise the takeoff speed. This greatly increases the drag on the wing though, requiring lots of power from the engines, so we cannot increase C_L too much. Landing is different. We need lots of drag to keep the plane from speeding up as it descends, so we can afford to put out all the flaps to increase both lift and drag. When it comes to arresting the descent and manoeuvring the plane at low speed on the approach, the pilot may occasionally have to apply full power just to keep the plane in the air. The sudden consequent increase in engine noise can sometimes be quite unnerving if you do not understand the reason for it.

Lift/drag ratio and angle of attack

Going back to equations (9) and (10) we can avoid the whole issue of what mass of air is affected as the aircraft flies along by eliminating m_{air}. This gives us

$$\mathrm{P_{ID}} = \tfrac{1}{2}\left(\frac{F_L}{v_{air}}\right) v_{air}^{\;2} = \tfrac{1}{2} F_L v_{air} \tag{16}$$

The power P_{ID} can be written as $F_{ID}\,v$ where F_{ID} is the drag force on the wing due to induced drag and v is the speed of the aircraft. Hence

$$F_{ID} = \frac{F_L v_{air}}{2v} \tag{17}$$

Now v_{air} is the vertical velocity of the downwash and it is reasonable to assume that, as long as the flow of air over the wing is laminar, the angle at which the air is deflected downwards by the wing is equal to the angle of attack of the wing α. Since this is a small angle we shall take tan α to be equal to α (in radians) so:

$$\frac{v_{air}}{v} = \alpha \tag{18}$$

from which, together with equations (1) and (4) we obtain the interesting result:

$$R_{LD} = \frac{F_L}{F_{ID}} = \frac{C_L}{C_{ID}} = \frac{2}{\alpha} \tag{19}$$

R_{LD} – the ratio of lift to drag - is a very important parameter. It gives a very good indication of the 'efficiency' of the wing and determines, for example, how far a glider or a bird can glide before it has to land. It is surprising, therefore, to see a formula for it which contains no reference to the area, shape or aspect ratio of the wing. It must be remembered that a highly efficient wing like that of an albatross or a glider will use a smaller angle of attack to generate a given lift force and will therefore have a greater lift/drag ratio.

Finally, by eliminating C_{ID} from equation (15) and (19) we arrive at a very useful theoretical result which relates the coefficient of lift C_L to the angle of attack α and the aspect ratio A:

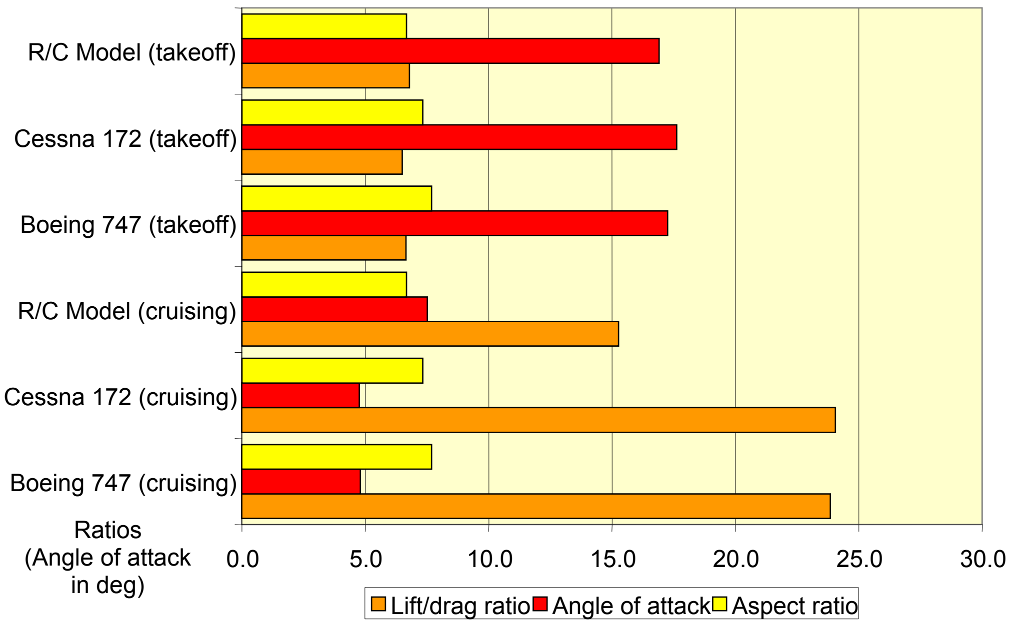
$$C_L = \frac{2\pi\,\alpha}{(1 + 2/A)} \tag{20}$$

For an aircraft with a very high aspect ratio, C_L tends towards $2\pi\alpha$ (α being in radians) or, for a convenient rule of thumb, C_L is approximately equal to the angle of attack in degrees divided by 10.

Real aircraft

It is time now to see how these formulae relate to the real world. For these illustrations I shall use data for the three aircraft mentioned earlier, a radio-conrolled model, a Cessna 172 and a Boeing 747 in each case on takeoff (without flaps) and cruising. *Fig 4.* compares various dimensionless ratios. All three aircraft have similar aspect ratios. On take-off, the angle of attack is high but in the cruise, attack angle decreases and lift/drag ratio increases. (The lift/drag ratio does not include parasitic drag.)

Fig. 4



Helicopters

Now let us apply the same principles to a helicopter while hovering in order to calculate the rotor frequency and the power required from the engine.

Consider a helicopter of mass M having a twin-bladed rotor each of length b, chord c and a lift coefficient C_L. The rotor blade is basically a wing which flies through the air. In order to calculate the total lift on the wing, we must take into account the fact that different parts of the wing are moving at different speeds. This means that we must integrate the lift on each small part of the wing along the length of the wing (ie from 0 to b for each blade). (For simplicity we will assume that C_L is constant along the wing.)

When rotating at a speed ω, a small element of the wing a distance r from the axis and of length dr has area $c.dr$ and is moving at a speed $v = r\omega$ so the wing has a total lift of

$$F_L = 2\int_0^b \tfrac{1}{2} C_L c \rho (r\omega)^2\, dr \tag{21}$$

$$= C_L\, c\, \rho\, \omega^2\, b^3 / 3 \tag{22}$$

Putting $\omega = 2\pi f$ (where f is the frequency of rotation) and $F_L = Mg$

$$Mg = 4\pi^2 C_L\, b^3\, c\, \rho\, f^2 / 3 \tag{23}$$

hence the rotor frequency f must be

$$f = \sqrt{\frac{3Mg}{4\pi^2 C_L b^3 c \rho}} \tag{24}$$

If we assume the following reasonable parameters for a small model helicopter: $M = 1.5$ kg, $b = 0.3$ m, $c = 0.04$ m and $C_L = 0.5$, this calculates to 40.3 Hz or about 2400 rpm. For a small full-sized helicopter such as the Huey ($M = 4500$ kg, $a = 14.6$ m, $b = 0.136$ m) the required rotor frequency is about 10 Hz (600 rpm).

Newton's Laws again

Now let us apply Newton's laws to the whole system in the following way. The effect of the rotors is to give the (assumed stationary) air above the rotor a mean vertical velocity downwards v_{air}. As before, the lift is simply the rate of change of momentum of the air and the power requirement is the rate of change of kinetic energy.

Now the mass of air passing through the rotor blades every second m_{air} is equal to the volume of air multiplied by its density. It might be thought that the volume of air passing through the rotor disc every second was equal to the area of the disc × v_{air}. This is not quite the case. Since the air is being accelerated from zero to v_{air} it is more appropriate to use the *average* speed, hence

$$m_{air} = \pi\, b^2 \rho \frac{v_{air}}{2} \tag{25}$$

This gives us the following relations:

$$\text{Lift} = m_{\mathrm{air}}\, v_{\mathrm{air}} = \pi\, b^2\, \rho\, v_{\mathrm{air}}^2 / 2 = Mg \tag{26}$$

And

$$\text{Power loss} = ½\, m_{\mathrm{air}}\, v_{\mathrm{air}}^2 = \pi\, b^2\, \rho\, v_{\mathrm{air}}^3 / 4 \tag{27}$$

Eliminating v_{air} from equations (26) and (27) leads us to the following important relation

$$\text{Power loss} = \sqrt{\frac{M^3 g^3}{2\pi\, b^2\, \rho}} \tag{28}$$

which tells us the minimum power required for any helicopter to hover. Using this formula, the model helicopter requires a power of 68 W while the Huey needs at least 460 kW (613 Hp)

(I say 'at least' because the velocity of the downwash will be greater at the tips than at the roots. This will increase the kinetic energy wasted. We are also ignoring any radial component of the downwash velocity.)

In both equations (24) and (28) the importance of having as large a rotor span as possible is transparently clear.

Wind Turbines

I have often wondered why wind turbines are designed with long, thin blades. Surely shorter but fatter blades like those of an office fan – or even the sails of an old-fashioned windmill – would 'capture more air' and produce more power? The answer is rather surprising.

Let us suppose, for the moment, that the wind is blowing at a speed v_{wind} and that the turbine is stationary. The blades are angled so as to produce a coefficient of lift $C_L = 1.5$ all along the blade. The total torque T on each blade will be:

$$T = \int_0^b ½\, C_{\mathrm{L}} c \rho\, v_{wind}{}^2\, r dr \tag{29}$$

which is simply

$$T = ¼ C_L\, b^2\, c\, \rho\, v_{wind}^2 \tag{30}$$

For a three-bladed wind turbine of blade length 40 m and mean chord 2 m in a 20 mph (9 m s^{-1}) wind, the stationary torque is 380,000 Nm.

Now when the blades turn, two things happen. First the angle at which the air strikes the blades becomes more oblique – it being the vector sum of the wind velocity and the blade velocity. This turns the lift force away from the desired direction and reduces the torque. On the other hand, the speed of the air flowing over the blade increases. It turns out that, by adjusting the pitch of the blades as the blades start to turn, the two effects cancel and the resultant torque remains approximately constant, independent of the speed of rotation. Since power = torque x angular velocity this would seem to imply that the power of a wind turbine is unlimited. Obviously this cannot be true, however. The discrepancy between theory and practice comes about because at a fast enough speed, each blade begins to encounter the downwash from the previous blade. In order to avoid this, the blades must be thin and spaced reasonably far apart, and the speed of the tips of the blades is limited to about 10 times the wind speed. This means that:

$$\omega = \frac{10\, v_{wind}}{b} \tag{31}$$

Using our figures, the turbine will be rotating at 2.25 rad s^{-1} (21 rpm) and will generate 853 kW. It is interesting to compare this figure with the total kinetic energy of the wind passing through the turbine disc every second. This is equal to $\frac{1}{2}\pi b^2 \rho v_{wind}^3$ and evaluates to 2.4 MW giving this turbine a very respectable efficiency of 35% (excluding losses due to parasitic drag, friction in the gearbox and electromagnetic losses, of course)

Conclusion

I hope that this article has convinced you that you do not need a degree in fluid dynamics to get an understanding of how aeroplanes fly, not to mention how helicopters and wind turbines work as well. I hope, too, to have opened your eyes to the educational possibilities of studying wings as part of an A level course on the applications of Newton's laws. Only a handful of A level physics syllabuses include any mention how aeroplanes fly and yet, surely, as you watch 499 other passengers with all their luggage boarding this massive machine which weighs a hundred times as much as a London bus, every thinking person must ask themselves – 'How on Earth is this thing going to get off the ground?". Now you know the answer. Newton's laws (and a b****y big engine)!

[1] Babinsky H 2003 How do wings work? *Phys. Educ.* **38** 497-503

An excel spreadsheet containing the data used to generate the charts illustrated above is available from the author. Please email jolinton@btinternet.com