The Lagrangian Points

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Abstract: There are five unique points in a star/planet system where a satellite can be placed whose orbital period is equal to that of the planet. Simple methods for calculating the positions of these points, or at least justifying their existence, are developed.

The Lagrangian Points L1-5

It is well-known that in the region of a mutually rotating pair of gravitating bodies such as a star and a planet there are 5 points at which a small satellite can exist in stable orbit[1] whose period is exactly the same as that of the planet. Three (L1, L2 & L3) lie on the line joining the two bodies while L4 and L5 are approximately 60 degrees ahead and behind the planet. (see *fig.1*)

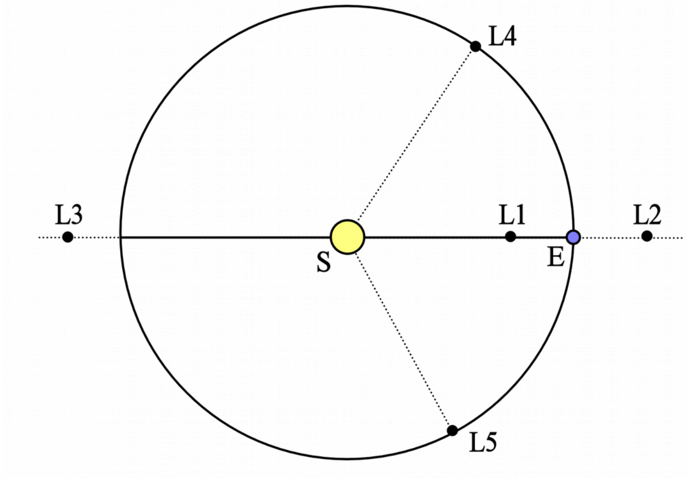

fig. 1

It is relatively easy to determine the positions of L1, L2 and L3.

Contrary to what you might expect, L1 is not at the point where the gravitational pull of the planet balances that of the star. It is at the point – slightly closer to the Sun – where the residual gravitational pull is just sufficient to cause a satellite placed there to orbit exactly once per year. (There are in fact several satellites there already one being SOHO which keeps a 24 hour watch on the Sun.) Another way of putting it is to say that the gravitational pull of the planet is used to 'slow down' the natural period of the satellite.

L2 and L3 are at the points outside the orbit of the planet where the extra gravitational pull of the planet is used to 'speed up' the satellite so that it keeps pace.

It is lot more difficult to explain why L4 and L5 exist and this is the main purpose of this article.

1 By 'stable' I mean that the forces on the satellite balance exactly. I do not imply that the orbit is stable under small perturbations.

Calculating the position of L1

Fig. 2 defines certain quantities.

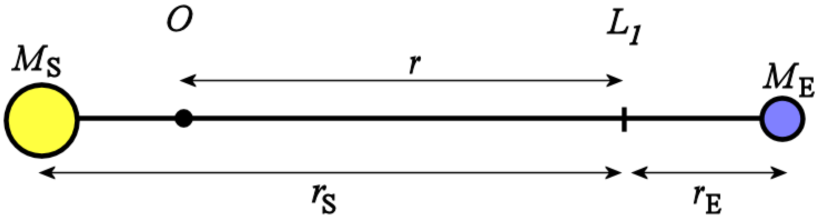

fig. 2

O is the common centre of mass of the system and this is also the centre of rotation of the system. In a system such as the Sun and the Earth where the star is so much more massive than the planet it can often be assumed that the centre of rotation coincides with the centre of the star but if we are going to calculate the position of the Lagrangian points accurately, we should not assume this. In fact the Sun is 330,000 times more massive than the Earth and therefore the common centre of mass lies about 453 km from the centre of the Sun. Since the radius of the Sun is 700,000 km this is obviously well inside the Sun itself[2] but, as we shall see when we come to consider L4 and L5, the discrepancy cannot be ignored.

It must also be remembered that the whole system depicted in the above illustration is rotating with angular velocity ω and that all the objects in the system are undergoing centripetal acceleration towards the centre of rotation of the system. For an object at L1 at a distance *r* from the centre of rotation, this acceleration is $r\omega^2$. We can now write down Newton's second law as follows:

$$\frac{GM_S m}{r_S^2} - \frac{GM_E m}{r_E^2} = mr\omega^2 \qquad (1)$$

Obviously the relation between r_S, r_E and *r* is complicated and solving the equation for *r* algebraically is not, in general, possible so an alternative method is to use a graphical technique. This involves introducing two new ideas. The first is familiar: use gravitational potentials instead of forces. Potentials are easier to handle than forces because they are scalar quantities and simply add up. The second idea is less familiar and uses the concept of a fictitious force.

Centrifugal force and Centrifugal potential

Suppose that the Sun and Earth were fixed to a huge merry-go-round. If you were to stand at the point *L*1 you would not only feel the gravitational attraction of the Sun and the Earth, you would also feel a force flinging you *outward* called centri*fugal* force. Normally we are at pains to claim that centrifugal force does not exist and that the only force acting on a satellite in orbit is a centri*petal* force but in a rotating frame of reference, fictitious forces like centrifugal force and Coriolis force really do exist and can be treated much like any other force.

Centrifugal force increases as you go away from the centre of rotation and obeys the following rule:

$$F_c = m\, r\omega^2 \qquad (2)$$

Now I have said that we are going to use potentials not forces so what does a centrifugal potential look like?

Well force is just minus a potential gradient so:

$$F_c = m\, r\omega^2 = -\frac{dV_c}{dr} \qquad (3)$$

2 This is not true of the Sun/Jupiter system where the centre of mass lies just outside the Sun

so by simple integration we get:

$$V_c = -\frac{1}{2} m r^2 \omega^2 \tag{4}$$

This is the equation of an inverted parabola. Think of it like an upside pudding-bowl. Place a marble on the centre and it will balance. Place it off centre and it will be flung outwards with increasing force just like a football on a rotating merry-go-round.

Now we can plot a graph of the total potential along a line between the Sun and the Earth. You can see what it looks like in *fig. 3*.

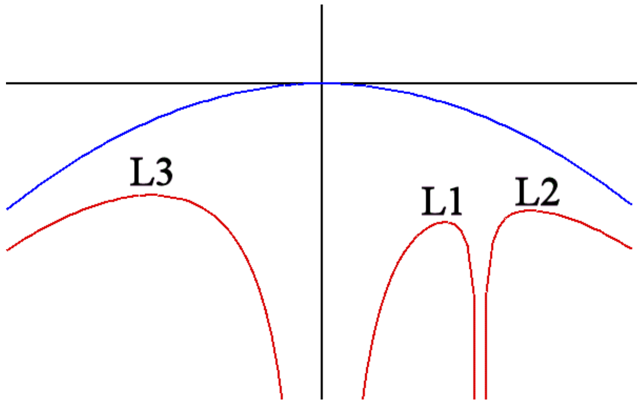

fig. 3

The blue line shows the 'centrifugal potential' and the red line shows the result of adding the two potential wells created by the star and the planet (not to scale, of course). It is immediately obvious that this curve has 3 points where the gradient is zero. These are the Lagrangian points LI, L2 and L3.

The equation of the red line is:

$$V_c = -\frac{GM_S m}{r_S} - \frac{GM_E m}{r_E} - mr^2\omega^2 \tag{5}$$

and while it is no easier to solve than equation (1) algebraically, it is a lot easier to interpret and can be readily solved by numerical methods.

L4 and L5

What about L4 and L5? At first sight it seems very unlikely that a satellite could travel in exactly the same orbit as the Earth because surely, it could be argued, the Earth would eventually attract the satellite and cause it to fall to earth.

What this argument fails to take into account is the fact, already mentioned, that the centre of rotation of the system does not coincide with the centre of the star. A satellite at L4 is acted upon by three forces: 1) the gravitational attraction of the Sun, 2) the gravitational attraction of the Earth and 3) (remembering that we are dealing with a rotating frame of reference) the 'centrifugal force' acting radially outwards from the centre of rotation. The crucial observation is that the attraction of the Sun and the outward centrifugal force are *not in the same line* and that there is a possibility that at some point these three forces might balance. The star S is pulling the satellite X ahead while the planet E is pulling it backwards (see *fig. 4*).

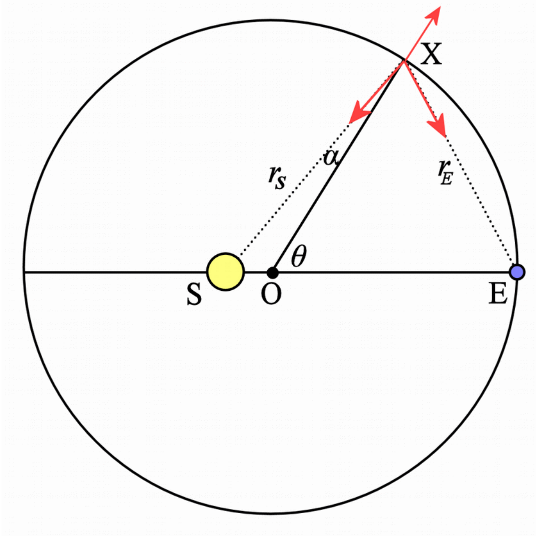

fig. 4

Now if the ratio of the mass of the Earth to that of the Sun is s (where s is a very small number) it is possible to show[3] that

$$\sin\alpha \approx \alpha \approx s\sin\theta \quad (6)$$

and that the tangential component of the gravitational attraction of the Sun (propelling the satellite forwards) is therefore equal to:

$$\frac{GM_S}{r_s^2} s\sin\theta \quad (7)$$

On the Earth side, the tangential component of the Earth's attraction pulling the satellite back is

$$\frac{GM_E}{r_e^2} \cos\theta/2 \quad (8)$$

The first thing to notice is that equation (7) contains the product of M_S and s. This is, of course equal to M_E. And the second thing to notice is that when $\theta = 60°$, r_S and r_E are approximately equal and, of course sin 60º is equal to cos 30º. In other words, when $\theta = 60°$ the forces on the satellite will balance!

3 The reason for this is as follows: In the triangle OSX we have $\frac{OS}{\sin\alpha}=\frac{XS}{\sin(180-\theta)}$ so $\sin\alpha = \frac{OS}{XS}\sin\theta$. But XS is very nearly equal to OX which equals OE and $\frac{OS}{OE}$ is, of course, equal to s.

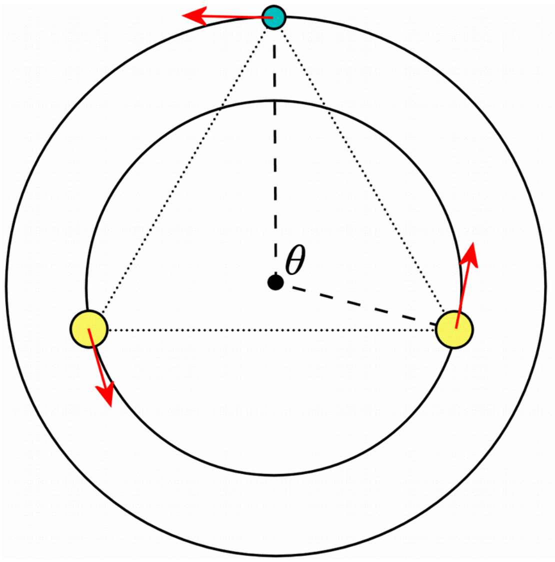

fig. 5

The above argument assumes that the mass of the planet is much less than that of the star, and that the mass of the satellite is negligible. What is truly remarkable, though, is that whatever the masses of the three bodies are, the three bodies S, E and X always form an exact equilateral triangle orbiting around their common centre of gravity! So, for example, a planet of considerable mass could orbit round a binary star system consisting of two stars of equal mass as shown in *fig. 5*. (It is worth noting, however, that the planet does not move in exactly the same orbit is the two stars and the angle of advance (θ) is no longer 60º.)

The potential map

Returning to the usual situation where the mass of the satellite is negligible, using potentials instead of forces gives us further insights into the situation. Using equation (5) in two dimensions generates the following potential map[4]:

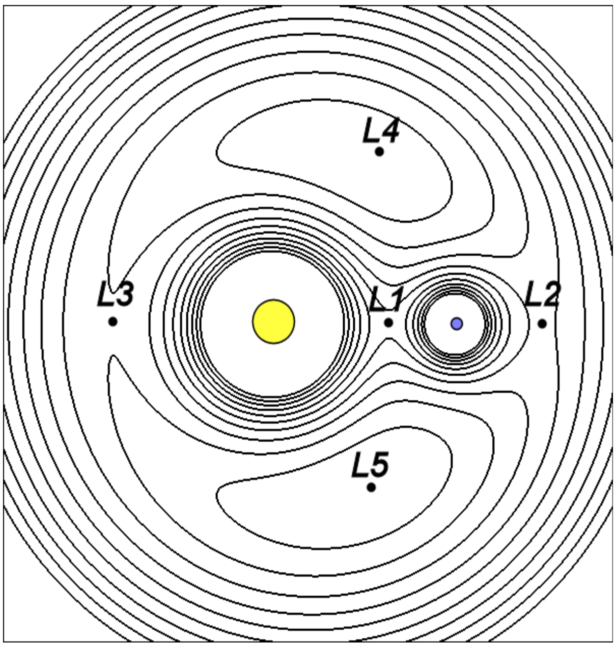

fig. 6

4 This map was drawn using a simple computer program written in BASIC which uses equation (5) to calculate the potential at every pixel in the image. A simple algorithm was then used to pick out the contour lines. Arbitrary values were used for the relative masses so as to emphasise the important features of the map.

One interesting difference is immediately apparent: LI, L2 and L3 are saddle points whereas L4 and L5 are local maxima. This has an important bearing on the stability of these orbits against perturbations. L1, L2 and L3 are unstable and a satellite such as SOHO placed at L1 needs constant monitoring and adjusting to make sure that it keeps on station. On the other hand, the Lagrangian points L4 and L5 are, in fact, stable against small perturbations. This may be good news for the scientists who wish to put satellites there but it has the disadvantage that they may have to share the space with a host of bits of dust and rocks which have accumulated there over the eons. Jupiter has at least 6000 'Trojan' asteroids at its Lagrange points which have actually been observed and, presumably, countless smaller ones. Earth has at least one known 'Trojan' and probably many more. The twin satellites STEREO A and STEREO B passed through Earth's L4 and L5 points some years ago on their way to their final destination at approximately ±120º with respect to the Earth.

Further information on Lagrangian Points and the satellites which use them can be found on Wikipedia (https://www.wikiwand.com/en/Lagrangian_point).

Some idea of the complexity of the mathematics required to prove the stability of L4 and L5 can be gained from a glance at http://www.math.cornell.edu/~templier/junior/final_paper/Thomas_Greenspan-Stability_of_Lagrange_points.pdf (just Google 'stability Lagrange points').

More information about the two STEREO satellites can be found at https://www.wikiwand.com/en/STEREO.