The art of guesstimation

J Oliver Linton

Old Mill Cottage, Crook, Kendal LA8 8LD

Email: jolinton@btinternet.com

Abstract. These days we are bombarded with figures in the media, many of which are of dubious authenticity and accuracy. The ability to make quick guesstimates of relevant data and to perform back-of-an-envelope calculations is a skill worth aquiring and therefore worth teaching.

‘*How many piano tuners are there in Chicago*?’

This is the most famous of the so-called ‘Fermi questions’. Enrico Fermi used questions like this to stimulate his students to think about how they could use their scientific knowledge to obtain approximate answers to almost any numerical question. Since then many teachers, particularly in America, have used the technique to improve their students’ numerical skills and many colleges and universities have used such questions to test potential students’ ability to think imaginatively and quickly.

I believe, however, that these questions have a much more important purpose and the ability to cope with questions like this is not merely a useful way of impressing your teacher and/or employer; it is an essential skill which every thinking citizen of this technologically obsessed civilisation should master. Consider the following statements all taken from a single recent (October 2008) newspaper:

- European nations spent £3.6bn on the LHC.
- A Bar-tailed Godwit has been recorded as flying 6,230 miles in 9 days.
- The Soleckshaw runs off a 36V battery that is topped up every 6 to 7 hours or 45 miles from a solar-powered charging station.
- Research done at Cambridge University shows that the release of more than a million tonnes of CO_2 could be avoided if we put the clocks back by 1 extra hour throughout the year.

The scientifically curious will immediately ask – just how much is £3.6bn? What is it as a fraction of Europe’s GDP, for example? Is it really possible for a bird to fly 6,000 miles in 9 days and if so, (and we must presume that the record is accurate since the bird was tagged) what implications does this remarkable fact have for the efficiency with which such a bird converts chemical energy into useful work? What area of solar panel is needed to charge up a 36V battery in a reasonable time? What is the justification for the statement about daylight-saving time and would 1 million tonnes of CO_2 be worth saving anyway?

In fact, every time we see an unsubstantiated figure in the newspaper we should be prepared to ask ourselves three important questions: I – is this figure feasible?; II – is this figure significant?; and III – what else can we deduce from this figure?

The skills needed to answer these questions are not trivial. I am not talking here about the ability to type the words ‘Europe GDP’ into Google or the ability to use a calculator and a precise formula; what I have in mind is the ability to quickly devise a strategy for working out an answer, make reasonable estimates of the relevant parameters and do approximate calculations on the back of an envelope preferably without a calculator. It is also important at the same time to be able to estimate the accuracy of your calculations. Sometimes an order of magnitude estimate will do; on other occasions you may need to aim for something better. Try it now with one of the questions listed above. I shall append my attempts to answer them at the end of this article.

To consider these strategies in more detail let us consider a more important question – how green is air travel as compared to other forms of transport such as car, bus or train?

The first step is to devise a strategy. It might be thought that it would be necessary to work out the amount of CO_2 emitted by each form of transport but since all forms of transport (except electric trains) use fossil fuels of one sort or another, it is sufficient to calculate the fuel consumption per passenger per unit distance. Let us start with the aeroplane. I know practically nothing about the fuel consumption of an aeroplane and I promise you that all of the figures below are genuine off-the-top-of-my-head estimates.

A typical wide-bodied jet carries 300 passengers and crew. Assuming that the average mass of a passenger is 80kg with 20kg of luggage, the total payload on the aircraft is 300×100 kg = 30 tonnes. Assuming that the aircraft's payload is between 10% and 30% of the aircraft's total mass, the mass of a typical aircraft (including passengers but excluding fuel) must be between 100 and 300 tonnes. Let's settle on 170 tonnes within a factor of about 1.8 either way.

(Note that when doing approximate calculations like these, the usual method of using % to indicate accuracy is useless because sometimes we may have to admit that our estimates may be more than 100% out! The best method to adopt when you have two widely spaced limits is to take the geometric mean of the limits and to indicate the accuracy of the estimate using a factor. I will use the notation '170 tonnes [×÷1.8]' to indicate the above estimate of the mass of an aeroplane.)

The next thing to do is to estimate the drag force which acts on the plane during the cruise. If the engines on an aircraft fail, its glide angle is fairly steep, probably between 5 and 10 degrees. This means that the drag force on an aeroplane is between 0.09 and 0.18 of its weight. Let's settle on 0.13 [×÷1.4]. The drag force on the plane must therefore be around $170{,}000 \times 10 \times 0.13 \approx 220{,}000$ N. [×÷2.5]. Over a distance of 1 km, the work done by the engines will therefore be 2.2×10^8 J [×÷2.5] (It is worth noting here that the accuracy factors must be multiplied together.)

Now there are, it must be admitted, a certain number of pieces of data which cannot be guessed but which every self-respecting scientists should carry around in their heads, and one of them is the calorific value of hydrocarbons. An approximate value can be found on the side of every packet of cornflakes and is about 1,600 kJ per 100g or 16 MJ/kg. The calorific value of fatty foods like butter is twice this and that of petroleum products, a little more. We shall not be far out if we use a figure of 40 MJ/kg.

Assuming that modern jet engines work close to their theoretical limit of efficiency of around 40%, the fuel needed to propel a modern jetliner a distance of 1 km through the air must be around $2.2 \times 10^8 / 0.4 / (40 \times 10^6) \approx 14$ kg with an error of not more that a factor of 2.5. i.e. somewhere between 6 and 35 kg per kilometre. (I will leave you to work out whether the climbing and descending phases of the flight make a significant difference to this figure.)

It would be wise at this point to ask ourselves at this point if this result is actually feasible and also to see if we could check our calculations by doing the estimate a completely different way. One might, for example, try to imagine how much fuel must be injected into the engines every second, say. It must surely be about the same as the rate of flow of water along a garden hosepipe. Now it takes about 20 s to fill a 5-litre watering can so this represents a flow of 0.25 litres/s. There are 4 engines on a modern plane (or 2 really large ones) and the density of fuel is around 0.8 kg/l so the total rate of fuel flow must be around 0.8 kg/s during which time the plane travels about 250 m. i.e. a fuel consumption rate of 3.2 kg/km.

So which of these two figures is correct? Our first estimate is at least based on some, admittedly rather shaky, physics. The second is no more than a guess – but the two answers are at least in the same ball park. As a further check, let's work out how much fuel a plane needs on a 10,000 km journey across the pond. Assuming a rate of consumption of 14 kg/km, the answer is 140 tonnes. This sounds a little bit too much for a plane which we assumed had a total (unfuelled) mass of 170 tonnes but it is certainly within a factor of 2 of the correct figure. Let's agree on 10 [×÷2] kg/km.

Finally we must relate the fuel consumption to the number of passengers. Assuming that the plane is full, the fuel consumption is 10 / 300 $\approx$ 0.033 [×÷2] kg/passenger-km.

Now what about a car? Fortunately the answer is well known as all car adverts carry this information. All we have to do is a bit of unit conversion. Let us take a modern family car which, if driven carefully, can do 50 [×÷1.2] miles to the gallon. 50 miles is 80 km and 1 gallon (4.5 litres) has a mass of around 3.5 kg. so the fuel consumption per passenger-km works out at 0.011 kg/passenger-km if the car is full or 0.045 kg/passenger-km if it contains only the driver. On this basis, a car is only more efficient than an aeroplane if it carries more than one person.

What about a bus? A bus which does 25 miles to the gallon and carries 20 passengers will be $(20/4) \times (25/50) = 2$ times more efficient than a car carrying 4 people.

Finally, the train. Curiously, this seems to be the most difficult estimate to make using either first principles or commonly available data but if we assume, as we did with the aeroplane, that the force needed to pull a train along at speed is something between 1/40th and 1/100th of its weight, the energy needed to pull a train of total mass 400 tonnes (100 tonnes for the loco and 300 tonnes for nine carriages) a distance of 1 km is between 40 and 100 MJ. Let us assume a figure of 65 MJ [×÷1.5] Assuming an engine efficiency of 30%, this will consume around 5.4 kg of fuel. If the train is full, it could be carrying as many as 500 passengers so the best fuel consumption figures work out to be around 0.01 kg/passenger-km. It is only the most popular trains that are full, however, and a more reasonable estimate of the average fuel consumption of a diesel powered passenger train would be at least double this and only about as good as a car carrying two people.

Our results can be graphed as shown in Figure 1.

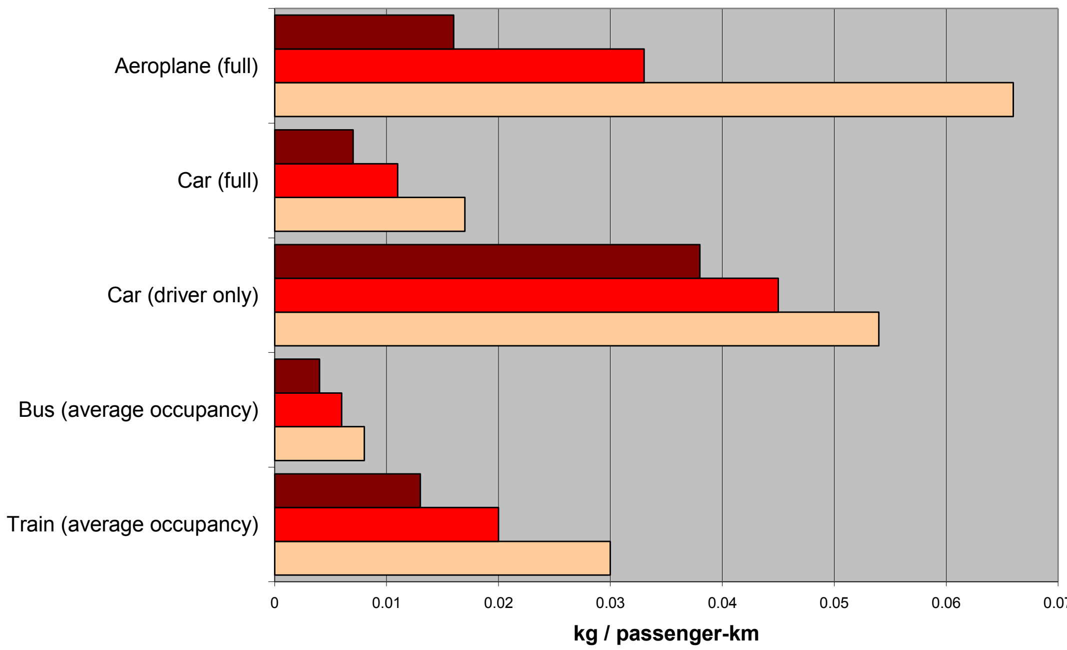

Figure 1. Fuel consumption of various modes of transport

The red bar is the guesstimated figure while the dark and light tan bars represent the assumed limits of accuracy.

I wonder if the results are what you expected. Perhaps the most surprising thing is not the differences between the figures but how similar they all are. Even the aeroplane, often accused of being the worst culprit when it comes to apportioning blame for global warming, competes favourably with the motor car, driven to and from work with one person aboard -though it has to be said that pollution in the upper atmosphere is probably more damaging to the environment than pollution near the ground.

Here are some more, equally important questions for you (or your students) to tackle:

- How much energy would be saved annually if everyone in the UK replaced their tungsten filament light bulbs with energy-saying light bulbs and how does this relate to the total amount of electrical energy consumed annually in the UK?
- How much energy is used to light our motorways at night and how does this compare with the answers to the first question above?
- How much CO_2 is breathed out annually by the whole human race and how does this compare with the amount of CO_2 produced annually by the burning of fossil fuels? How does this compare with the mass of the Earth's atmosphere?
- How many wind turbines would be needed to supply all of the UK's electricity needs?
- It has been said that a 1^o rise in global temperature could cause a 3 m rise in sea levels. How much of this rise can be attributed to the expansion of sea water?

I am sure you can think of many more such question but if you run out of ideas, the following book will supply them: **Weinstein and Adam** *Guesstimation* Princeton University Press ISBN 978-0-691-12945-5

My answers to the four questions posed at the beginning of this article are as follows:

- £3.6bn is approximately 0.1% of the total amount earned by 200 million people earning an average of £20,000 per year. Money well spent, I would say!
- Making similar assumptions to those used in calculating the fuel consumption of an aeroplane, I estimate that the bird must have used up at least one third of its body mass on the journey. That is quite some work-out!
- It is not immediately obvious what factors the researchers at Cambridge University took into account when calculating the expected savings due to extending the period of daylight saving time and the newspaper did not say. At first sight it would appear to make little difference to our consumption of energy as what you gain at one end you lose at the other – after all, life must go on. There will, however be an effect in the summer months because the sun rises much earlier than most people. Assuming, therefore that we can save 100W per person for 1 hour at the end of the day for 6 months of the year, we can save 1 TW-hour every year. To generate this electricty we need 300,000 tonnes of coal which, when burned, produces 1.1 million tonnes of CO_2. So the figure is feasible. Is it worth saving? I will leave that to you.
- The energy stored in a 50 A-h, 36V battery is 1800 W-h. Assuming a charging period of 9 hours, the battery must be charged at a rate of at least 200W. Assuming that a solar panel has an efficiency of 10% and that the solar constant at the surface of the Earth is 1 kW (This is another figure that every scientist should know) the panel will have to have an area of at least 2 m^2.

What were your guesstimates?