Complex Power Series

The geometric progression

Let $S = 1 + x + x^2 + x^3 + \ldots$

Then $S\,x = x + x^2 + x^3 + \ldots = S - 1$

Hence $S\,x = S - 1$

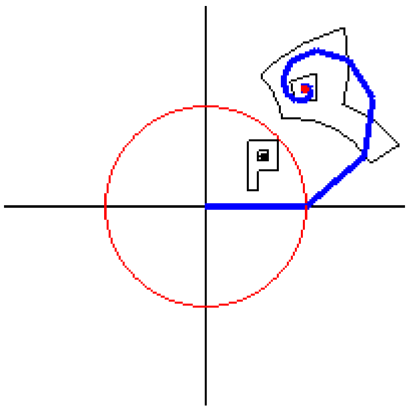
and $S = \dfrac{1}{1 - x}$

In general, where z is a complex number

$$\frac{1}{1 - z} = 1 + z + z^2 + z^3 + z^4 + \ldots$$

If we put $z = r\,e^{i\theta}$ we get the series $1 + r\,e^{i\theta} + r^2 e^{2i\theta} + r^3 e^{3i\theta} + \ldots$. It is obvious that this series will only converge if $r < 1$; ie if z lies within the unit circle.

A typical summation looks like this:



Replacing z with $-z$ gives us the series:

$$\frac{1}{1 + z} = 1 - z + z^2 - z^3 + z^4 - \ldots$$

which is the same mapping rotated 180° about the origin.

Replacing z with z^2 gives us

$$\frac{1}{1 - z^2} = 1 + z^2 + z^4 + z^6 + \ldots$$

which is similar.

The binomial series

We want to show that

$$(1+z)^p = 1 + \frac{p}{1!}\,z + \frac{p(p - 1)}{2!}\,z^2 + \frac{p(p - 1)(p - 2)}{3!}\,z^3 + \ldots$$

We notice that the n^{th} coefficient of the expansion is equal to ${}^{p}C_n$ – (ie the number of ways in which n objects can be drawn from a pile of p objects). Now it is a defining characteristic of the coefficients of the Binomial Series that each coefficient ${}^{p}C_n$ is the sum of the two coefficients in the expansion of $(1+z)^{p-1}$ ie ${}^{p}C_n = {}^{p-1}C_{n-1} + {}^{p-1}C_n$. We must therefore show that

$$\frac{p(p-1)\ldots(p-(n-1))}{1.2.3\ldots n} \text{ is equal to } \frac{(p-1)\ldots(p-(n-1))}{1.2.3\ldots n-1} + \frac{(p-1)\ldots(p-n)}{1.2.3\ldots n}$$

We have

$$\frac{(p-1)\ldots(p-(n-1))}{1.2.3\ldots n-1} + \frac{(p-1)\ldots(p-n)}{1.2.3\ldots n} =$$

$$\frac{n(p-1)\ldots(p-(n-1)) + (p-1)\ldots(p-n)}{1.2.3\ldots n} =$$

$$\frac{n(p-1)\ldots(p-(n-1)) + (p-n)(p-1)\ldots(p-(n-1))}{1.2.3\ldots n} =$$

$$\frac{((p-1)\ldots(p-(n-1))).(n+(p-n))}{1.2.3\ldots n} =$$

$$\frac{(p(p-1)\ldots(p-(n-1)))}{1.2.3\ldots n} \quad \text{Q.E.D.}$$

[As an example, take p = 7 and n = 4

$$\frac{7.6.5.4}{1.2.3.4} = 35$$

The two preceding coefficients are

$$\frac{6.5.4.3}{1.2.3.4} = 15 \text{ and } \frac{6.5.4}{1.2.3} = 20$$

I find it remarkable that the addition of the extra n (here equal to 4) needed to make the denominators equal - to the spare number in the numerator (here equal to 3) - is exactly what is needed to make the front end for the new larger factorial (here equal to 7)!]

Technically, all we have to do now is to show that the formula works for n and p equal to 1 but this is trivial.

The exponential function

We want to show that

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

Let us start by considering what happens to the expression $(1 + 1/n)^n$ as n tends to ∞. Using the Binomial Theorem we get

$$\left(1+\frac{1}{n}\right)^n = 1 + \frac{n}{1}\left(\frac{1}{n}\right)^1 + \frac{n(n-1)}{1.2}\left(\frac{1}{n}\right)^2 + \ldots$$

Now if n is really large, the factorial terms in the numerator of each coefficient are cancelled out by the powers of n in the denominator leaving just the numeric factorials on the bottom. In other words:

$$\lim\left(1+\frac{1}{n}\right)^n \text{ as } n\rightarrow\infty = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\ldots = 2.7182818285\ldots$$

Obviously this is a very interesting number. We shall call it *e*.

Now let us consider $\lim \left(1 + \frac{1}{n}\right)^{n\,z}$ as $n \to \infty$. It is a fact (though not necessarily an obvious one) that this will be equal to e^z.

First we have

$$\left(1 + \frac{1}{n}\right)^{n\,z} = 1 + \frac{n}{1}\left(\frac{1}{nz}\right)^1 + \frac{n(n-1)}{1.2}\left(\frac{1}{nz}\right)^2 + \ldots$$

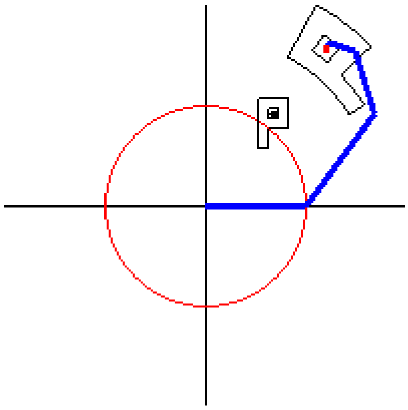
and in the limit we get the very important series expansion:

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

It is worth noting here that if you differentiate this series, each term turns into the one that precedes it. It is therefore a defining characteristic of the function e^Z that it is its own derivative ie:

$$\frac{\mathrm{d}(e^z)}{\mathrm{d}z} = e^z \ .$$

Since the factorial function increases faster than any polynomial, the exponential series converges for all values of *z*. A typical summation is shown below:



It is interesting to note that horizontal lines are transformed into radial lines and vertical ones into circles about the origin. Alternatively we can say that the real part of *z* becomes the radius (actually $r = e^{(\text{real } z)}$ and the imaginary part becomes the argument.

The trigonometric functions

We want to show that

$$\sin(z) = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

We have seen that the exponential function is its own derivative and that we can, in fact, define it as such. We can derive expansions of the trigonometrical functions in a similar way.

It is a defining characteristic of the sin and cos functions

that when they are differentiated twice, they tun into negatives of themselves. eg:

$$\frac{d^2(\sin z)}{dz^2} = -\sin(z)$$

If we assume that $\sin(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$ we can see at once by putting $z = 0$ that a_0 must be equal to 0 and by differentiating once and putting $z = 0$, a_1 must be equal to 1.

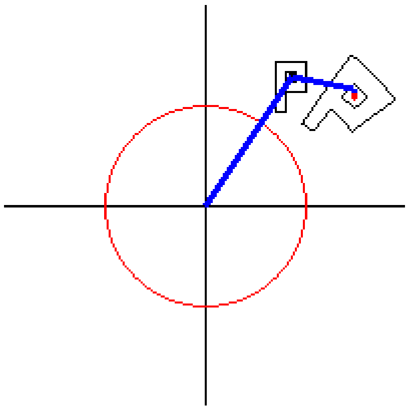
By differentiating twice we get

$\sin(z) = -1.2a_2 + -1.2.3 a_3 z + -1.2.3.4 a_4 z^2 + -1.2.3.4.5 a_5 z^3 + \ldots$ in which case a_2 must be equal to $-a_0 / 2 = 0$, a_3 must be equal to $- a_1 /3!$, a_4 must be equal to 0, a_5 must be equal to $+ a_1 /5!$ etc.

A similar argument holds for cos z.

As with the exponential function, the series are convergent for all values of z

A typical summation is shown below:



If we put $z = i\theta$ we get $e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \ldots$

We can see at once therefore that $\cos\theta + i\sin\theta = e^{i\theta}$

Putting $z = -i\theta$ we get $e^{i\theta} = 1 - \frac{i\theta}{1!} - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} + \ldots$

in which case $\cos\theta - i\sin\theta = e^{-i\theta}$

Combining these equations gives us $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

The hyperbolic functions

We want to show that

$$\sinh(z) = \frac{z}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \ldots$$

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots$$

It is a defining characteristic of the sinh and cosh functions that each one differentiates into the other. This can be achieved simply by taking alternate terms from the expansion of e^z.

This leads us to the expressions $\cosh\theta + \sinh\theta = e^{\theta}$, $\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$ and $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

Taylor's theorem

Before deriving any more series, we need to derive Taylor's theorem

suppose that a function $f(z)$ can be expanded as a power series ie:

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

It is obvious that a_0 is simply the value of the function at $z = 0$

Differentiating once we get:

$$f'(z) = a_1 + a_2 z + 3a_3 z^2 + 4a_4 x^3 + \ldots$$

from which we see that a_1 is the value of $f'(z)$ at $z = 0$

In general, after differentiating n times, the value of a_n is the value of $f^n(z)$ at $z = 0$ and this term is equal to $n!$ Hence:

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \ldots$$

This is known as McLaurin's theorem.

(It seems to me remarkable that if you know all the derivatives of a function at the origin, you can predict the value of the function anywhere else. Of course, this is all due to the fact that we have restricted ourselves to functions which can be expanded as a power series. Clearly Taylor's theorem cannot be applied to a function like $x = x \bmod 2$.)

(It is worth noting here that since all the derivatives of e^z are unity at $z = 0$, the expansion of e^z follows at once.)

There in nothing particularly special about the origin. We can, instead, use the derivatives at any point by expanding the function in terms of powers of $z - c$. Hence if we assume that

$$f(z) = a_0 + a_1(z - c) + a_2(z - c)^2 + a_3(z - c)^3 + \ldots$$

this leads us to Taylor's theorem:

$$f(z) = f(c) + \frac{f'(c)}{1!}(z - c) + \frac{f''(c)}{2!}(z - c)^2 + \frac{f'''(c)}{3!}(z - c)^3 + \ldots$$

Alternatively, putting $c = 1$ and replacing z by $1 + z$,

$$f(1+z) = f(1) + \frac{f'(1)}{1!} z + \frac{f''(1)}{2!} z^2 + \frac{f'''(1)}{3!} z^3 + \ldots$$

As a trivial example consider the exponential function. At $z = 1$, all the derivatives are equal to e

hence $e^{(1 + z)} = e + \dfrac{e}{1!} z + \dfrac{e}{2!} z^2 + \dfrac{e}{3!} z^3 + \ldots = e(e^z)$

The binomial series (again)

We want to show that

$$(1+z)^p = 1 + \frac{p}{1!} z + \frac{p(p-1)}{2!} z^2 + \frac{p(p-1)(p-2)}{3!} z^3 + \ldots$$

At $z = 0$, $f(z) = 1$

Differentiating once, $f'(z) = p(1+z)^{(p-1)}$ so $f'(0) = p$

Differentiating again, $f''(z) = p(p-1)(1+z)^{(p-2)}$ so $f''(0) = p(p-1)$

In general $f^n(z) = p(p-1)(\ldots)(p-n+1)(1+z)^{(p-n)}$ $:$ $f^n(0) = p(p-1)(\ldots)(p-n+1)$

Using McLaurin's theorem therefore we obtain the above identity.

The logarithmic function

We want to show that:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$$

It is a defining characteristic of the (natural) logarithm function that $\frac{\mathrm{d}\log(x)}{\mathrm{d}x} = \frac{1}{x}$. At $x = 0$, the gradient is infinite and so cannot be expanded as a simple power series. Instead we examine the function $\log(1 + x)$, or, more generally, $\log(1 + z)$.

Now at $z = 0$. $\log(1 + z) = \log(1) = 0$ so the first coefficient of our expansion is 0

We have $\frac{\mathrm{d}\log(1+z)}{\mathrm{d}z} = \frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$

so the second coefficient is 1

Differentiating again we get $\frac{\mathrm{d}^2\log(1+z)}{\mathrm{d}z^2} = -1 + 2z - 3z^2 + \ldots$

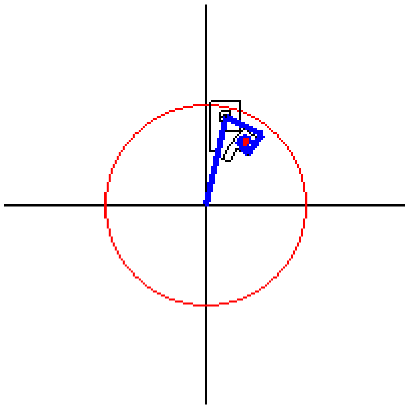
so the third coefficient is -1

Successive differentiating give us the series of coefficients 0, 1, -1, 2.1, -3.2.1, 4.3.2.1 ...

The n^{th} coefficient is $-(-1)^n(n-1)!$ Hence

$$\log(1+z) = 0 + \frac{1}{1!}z + \frac{-1}{2!}z^2 + \frac{2!}{3!}z^3 + \frac{-3!}{4!}z^4 + \ldots$$
$$= 0 + z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \ldots$$

Note that if we put $z = 1$, we obtain the alternating harmonic series which is not absolutely convergent. This means that the above series is only convergent within the unit circle.



A series for π

Consider the expansion of the expression log(1 + *i* tan*θ*)

$$\log(1 + i\tan\theta) = i\tan\theta + \frac{\tan^2\theta}{2} - i\frac{\tan^3\theta}{3} - \frac{\tan^4\theta}{4} + i\frac{\tan^5\theta}{5} + \dots$$

Now

$$\log(1 + i\tan\theta) = \log\left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right) = \log\left(\frac{e^{i\theta}}{\cos\theta}\right) = i\theta - \log(\cos\theta)$$

So what? You might say. Well the interesting bit is the imaginary part which, if we extract it gives us a series for θ in terms of powers of tanθ. (We are more used to seeing expansions of tanθ in terms of θ.)

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \frac{\tan^7\theta}{7} + \dots$$

We could, of course write this as

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots$$

Now we know that when $\theta = \pi/4$, $\tan\theta = 1$ so:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Alternatively, we can use $\theta = \pi/6$, $\tan\theta = 1/\sqrt{3}$

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3.3} + \frac{1}{5.3.3} - \frac{1}{7.3.3.3} + \dots\right)$$

which converges a bit more quickly but requires the calculation of a square root