Hypercubes and Hyperspheres

Hypercubes

To turn a point into a line, you double the number of points and add a line giving 2P, 1L

To turn a line into a square you double the original line and pull out each point into a new line. You also create a new area giving a total of 4P, 4L, 1A

To turn a square into a cube you double the original square and pull out each point into a new line. Each of the original lines in the square becomes a new area and the original area becomes a new volume; hence a cube has 8P, 12L, 6A and 1V

It is easy to generalise this series. If a certain h-Cube has n_0 points, n_1 lines, n_2 areas etc then the next h-Cube up will have $2n_0 + 0$ points, $2n_1 + n_0$ lines, $2n_2 + n_1$ areas etc

Each cell in the spreadsheet below is the sum of 2 × the previous cell (on the left) plus the one above that.

	n	Point	Line	Square	Cube	4-cube	5-cube	6-cube	7-cube	8-cube	9-cube	10-cube
	0	0	0	0	0	0	0	0	0	0	0	0
Points	0	1	2	4	8	16	32	64	128	256	512	1024
Lines	1		1	4	12	32	80	192	448	1024	2304	5120
Areas	2			1	6	24	80	240	672	1792	4608	11520
Volumes	3				1	8	40	160	560	1792	5376	15360
4-volumes	4					1	10	60	280	1120	4032	13440
5-volumes	5						1	12	84	448	2016	8064
6-volumes	6							1	14	112	672	3360
7-volumes	7								1	16	144	960
8-volumes	8									1	18	180
9-volumes	9										1	20
10-volumes	10											1

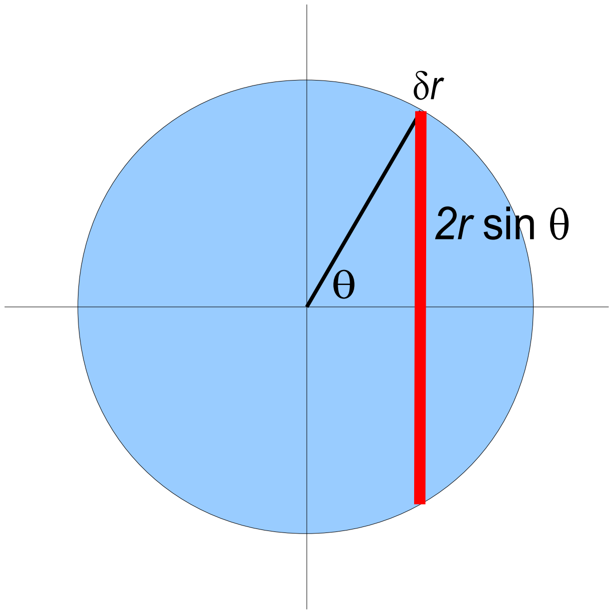
Hyperspheres

To turn a turn a double unit line into a circle, you double the line and 'blow it up' into a circle. the line increases in length by a factor π and an area is created equal to π.

To turn a circle into a sphere you double the circle and 'blow it up' into a sphere. The area increases by a factor of 4 and a volume is created equal to $4/3\ \pi$.

To turn a sphere into a hypersphere you double the sphere and 'blow it up' but it is not easy to visualised the result – nor is it easy to guess the factors by which the volume is increased or the new hyper-volume which is created.

To calculate the area of a circle you have to integrate of a whole lot of lines. eg:



$$A = \int_{-r}^{r} 2r\sin\theta \, \mathrm{d}r = \int_{0}^{\pi} 2\mathrm{r}\sin\theta . r\sin\theta \, \mathrm{d}\theta = r^2 \int_{0}^{\pi} (1-\cos 2\theta) \, \mathrm{d}\theta = \pi r^2$$

To find the volume of a sphere, you must integrate a lot of circles (the same diagram will do)

$$V = \int_{-r}^{r} \pi (r\sin\theta)^2 \mathrm{d}r = \int_{0}^{\pi} \pi (r\sin\theta)^2 . r\sin\theta \, \mathrm{d}\theta = \frac{r^3}{4} \int_{0}^{\pi} (3\sin\theta - \sin 3\theta) \, \mathrm{d}\theta = \frac{4}{3} \pi r^2$$

To find the volume of a hypersphere, you simply integrate a whole lot of spheres. The integrals get pretty tedious so here is a summary of the results:

	n-Area	*n-Volume*
Circle	$2 \pi r$	πr^2
Sphere	$4 \pi r^2$	$4/3 \pi r^3$
4-sphere	$2 \pi^2 r^3$	$1/2 \pi^2 r^4$
5-sphere	$8/3 \pi^2 r^4$	$8/15 \pi^2 r^5$
6-sphere	$\pi^3 r^5$	$1/6 \pi^3 r^6$

In general if V_n is the coefficient of the n-Volume and A_n is the coefficient of the surface n-Area.

$$V_n = I_n V_{n-1} \quad \text{and} \quad A_n = n V_n$$

where I_n is the integral $\int_{0}^{x} \sin^x \theta \, \mathrm{d}\theta$