Lagrangians and Hamiltonians

The Principle of least action

There is an extremely deep principle called the principle of least action which governs all of physics. It is deeper than the conservation laws of energy and momentum because, together with some fundamental assumptions about the isotropy of space and time, these can be derived from it and basically what it says is that when a system evolves from a state A to a state B it does so in a way which minimises a certain quantity called **action** $\boldsymbol{A}$. Now action is the integral with respect to time of a quantity which has the dimensions of energy which is called the **Lagrangian** $\boldsymbol{L}$. so:

$$A = \int_{t_0}^{t_1} L\,dt \tag{1}$$

and for a single particle moving about in a potential field in one dimension (eg a rollercoaster) the Lagrangian is equal to its kinetic energy **minus** its potential energy. i.e.:

$$L = KE - PE = \frac{1}{2} m \dot{x}^2 - V(x) \tag{2}$$

N.B. The Lagrangian is a function of both x and $\dot{x}$ so we can differentiate it (partially) with respect to either x or $\dot{x}$

$\frac{\partial L}{\partial x}$ is a measure of the amount by which L changes along the roller coaster solely by virtue of changes in potential energy. Similarly $\frac{\partial L}{\partial \dot{x}}$ is a measure of the way L changes because of changes in speed, regardless of the height of the roller coaster.

The Euler-Lagrange Equation

Now Euler and Lagrange worked out that for action to be minimised the following relation must always hold:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}} \tag{3}$$

and by using this piece of mathematical magic, we can work out the equations of motion of a roller coaster. Watch.

From equation (2) we have:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

and

$$\frac{\partial L}{\partial x} = -\frac{d\,V(x)}{dx}$$

so from equation (3) we get

$$m\ddot{x} = -\frac{dV(x)}{dx}$$

which tells us that where the roller coaster comes across a steep ascent, the train will decelerate and on a descent it will accelerate in accordance with Newton's laws of motion!

Lets see if we can work out the equations of motion of a planet in orbit round a sun. The beauty of the Euler-Lagrange equation is that we are not restricted to Cartesian coordinates. We can use any coordinate system that we like and there is an Euler-Lagrange equation for each coordinate. Obviously we should use polar coordinates here. Watch:

First the kinetic energy:

$$KE = 1/2m((r\dot{\theta})^2 + \dot{r}^2)$$

Now the potential energy:

$$PE = -GMm/r$$

so

$$L = 1/2m((r\dot{\theta})^2 + \dot{r}^2) + GMm/r \tag{4}$$

Now the two partial differentials in *r*:

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ and } \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - GMm/r^2$$

from which we get

$$\ddot{r} = r\dot{\theta}^2 - GM/r^2 \tag{5}$$

The first term is, of course, the centrifugal acceleration (it is centri*fugal* because in the absence of gravity *r* would *increase*) and the second term is the acceleration due to gravity.

We can also calculate the partial differentials in θ

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \text{ and } \frac{\partial L}{\partial \theta} = 0$$

from which we get

$$\frac{d(mr^2\dot{\theta})}{dt} = 0 \tag{6}$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} \tag{7}$$

This tells us that when the planet is travelling towards the sun (ie $\dot{r}$ is negative) the rotational speed will *increase*. In fact, as equation (6) shows, it is telling us that the angular momentum remains constant.

What about a simple pendulum?

This is easy. There is only one variable θ.

$$KE = 1/2m(l\dot{\theta})^2$$

$$PE = mgl(1 - \cos\theta)$$

$$L = 1/2m(l\dot{\theta})^2 - mgl(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \text{ and } \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$2ml\dot{l}\dot{\theta} + ml^2\ddot{\theta} = -mgl\sin\theta$$
$$2\dot{l}\dot{\theta} + l\ddot{\theta} = -g\sin\theta \tag{8}$$

and that's it! That's the equation of motion for a simple pendulum – even one which rotates round and round. OK, solving this equation (ie finding an analytical solution) is extremely difficult and in many cases impossible but it is easy to solve numerically. But getting to equation (8) using Newton's laws is a lot more difficult.

What about a double pendulum? Lets restrict ourselves to the case where two masses each of mass 1 unit are linked by rods of length 1 unit. We shall use two coordinates to define the system: θ will be the angle the first rod makes with the vertical and α will be the angle between the first rod and the second. (The expressions for KE and PE are not difficult to prove but I have cheated and used Susskind's results!)

$$KE = \dot{\theta}^2 + (\dot{\theta} + \dot{\alpha})^2/2 + \dot{\theta}(\dot{\theta} + \dot{\alpha})\cos\alpha$$
$$PE = -g(2\cos\theta + \cos(\theta - \alpha))$$
$$L = \dot{\theta}^2 + (\dot{\theta} + \dot{\alpha})^2/2 + \dot{\theta}(\dot{\theta} + \dot{\alpha})\cos\alpha + g(2\cos\theta + \cos(\theta - \alpha))$$
$$\frac{\partial L}{\partial\dot{\theta}} = 3\dot{\theta} + \dot{\alpha} + (2\dot{\theta} + \dot{\alpha})\cos\alpha$$
$$\frac{\partial L}{\partial\theta} = -3g\sin\theta$$
$$3\ddot{\theta} + \ddot{\alpha} + (2\dot{\theta} + \alpha)\sin\alpha\,\dot{\alpha} + (2\ddot{\theta} + \ddot{\alpha})\cos\alpha = -3g\sin\theta \tag{9}$$
$$\frac{\partial L}{\partial\dot{\alpha}} = \dot{\theta} + \dot{\alpha} + \dot{\theta}\cos\alpha$$
$$\frac{\partial L}{\partial\alpha} = -\dot{\theta}(\dot{\theta} + \dot{\alpha})\sin\alpha + g\sin\alpha$$
$$\ddot{\theta} + \ddot{\alpha} - \dot{\theta}\sin\alpha\,\dot{\alpha} + \ddot{\theta}\cos\alpha + \dot{\theta}(\dot{\theta} + \dot{\alpha})\sin\alpha = g\sin\alpha \tag{10}$$

and from equations (9) and (10) you can work out expressions for $\ddot{\alpha}$ and $\ddot{\theta}$