Linear Transformations

Transformations

A linear transformation may be defined as follows:

$$\begin{aligned} x' &= ax + by + c \\ y' &= dx + ey + f \end{aligned}$$

Certain special cases stand out. This is a translation:

$$\begin{aligned} x' &= x + c \\ y' &= y + f \end{aligned}$$

This is a scaling:

$$\begin{aligned} x' &= ax \\ y' &= ey \end{aligned}$$

This is a rotation:

$$\begin{aligned} x' &= \cos(\theta)x - \sin(\theta)y \\ y' &= \sin(\theta)x + \cos(\theta)y \end{aligned}$$

It is obvious that two successive translations, or two successive scalings, are equivalent to a single transformation obtained by adding the respective translations or multiplying the scalings. It is not so clear what happens when two successive rotations are applied. Suppose that:

$$\begin{aligned} x' &= \cos(\theta)x - \sin(\theta)y \\ y' &= \sin(\theta)x + \cos(\theta)y \end{aligned}$$

and

$$\begin{aligned} x'' &= \cos(\phi)x' - \sin(\phi)v' \\ y'' &= \sin(\phi)x' + \cos(\phi)y' \end{aligned}$$

eliminating x' and y' we get

$$\begin{aligned} x'' &= \cos(\phi)(\cos(\theta)x - \sin(\theta)y) - \sin(\phi)(\sin(\theta)x + \cos(\theta)y) \\ y'' &= \sin(\phi)(\cos(\theta)x - \sin(\theta)y) + \cos(\phi)(\sin(\theta)x + \cos(\theta)y) \end{aligned}$$

hence

$$\begin{aligned} x'' &= (\cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta))x - (\cos(\phi)\sin(\theta) + \sin(\phi)\cos(\theta))y \\ y'' &= (\sin(\phi)\cos(\theta) + \cos(\phi)\sin(\theta))x + (\cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta))y \end{aligned}$$

which is, of course,

$$\begin{aligned} x'' &= \cos(\phi + \theta)x - \sin(\phi + \theta)y \\ y'' &= \sin(\phi + \theta)x + \cos(\phi + \theta)y \end{aligned}$$

proving that two successive rotations are equivalent to a single rotation through the sum of the two angles.

What about a rotation followed by a translation? We have:

$$\begin{aligned} x' &= \cos(\theta)x - \sin(\theta)y \\ y' &= \sin(\theta)x + \cos(\theta)y \end{aligned} \quad \text{and} \quad \begin{aligned} x'' &= x' + c \\ y'' &= y' + f \end{aligned}$$

from which we get:

$$\begin{aligned} x'' &= \cos(\theta)x - \sin(\theta)y + c \\ y'' &= \sin(\theta)x + \cos(\theta)y + f \end{aligned}$$

But what about a translation followed by a rotation? This time we have:

$$\begin{aligned} x' &= x + c \\ y' &= y + f \end{aligned} \quad \text{and} \quad \begin{aligned} x'' &= \cos(\theta)x' - \sin(\theta)y' \\ y'' &= \sin(\theta)x' + \cos(\theta)y' \end{aligned}$$

from which we get:

$$\begin{aligned} x'' &= \cos(\theta)(x + c) - \sin(\theta)(y + f) \\ y'' &= \sin(\theta)(x + c) + \cos(\theta)(y + f) \end{aligned}$$

or

$$\begin{aligned} x'' &= \cos(\theta)x - \sin(\theta)y + (c\cos(\theta) - f\sin(\theta)) \\ y'' &= \sin(\theta)x + \cos(\theta)y + (c\sin(\theta) + f\cos(\theta)) \end{aligned}$$

which is not the same at all. Clearly the order in which transformations take place is sometimes significant.

Matrices

The mechanics of combining transformations can be greatly streamlined by the use of matrices. Two matrices can only be multiplied together if one has the same number of columns as the other has rows. Since we want all our matrices to look the same, we must use square 3x3 matrices. The multiplication of two 3x3 matrices looks like this:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} =$$

$$\begin{bmatrix} A_{11}B_{11}+A_{12}B_{21}+A_{13}B_{31} & A_{11}B_{12}+A_{12}B_{22}+A_{13}B_{32} & A_{11}B_{13}+A_{12}B_{23}+A_{13}B_{33} \\ A_{21}B_{11}+A_{22}B_{21}+A_{23}B_{31} & A_{21}B_{12}+A_{22}B_{22}+A_{23}B_{32} & A_{21}B_{13}+A_{22}B_{23}+A_{23}B_{33} \\ A_{31}B_{11}+A_{32}B_{21}+A_{33}B_{31} & A_{31}B_{12}+A_{32}B_{22}+A_{33}B_{32} & A_{31}B_{13}+A_{32}B_{23}+A_{33}B_{33} \end{bmatrix}$$

In general, to obtain the term in the r^{th} row of the c^{th} column of the result, we sum the products of the terms of the r^{th} row of the first matrix with those of the c^{th} column in the second.

The matrix for a translation looks like this:

$$\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$$

and a rotation:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Notice the 1 in the bottom right hand corner!)

If we multiply these together we get:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & c \\ \sin(\theta) & \cos(\theta) & f \\ 0 & 0 & 1 \end{bmatrix}$$

which is indeed the matrix for a rotation followed by a translation.

What about multiplying them the other way round? This time we get:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & c\cos(\theta) - f\sin(\theta) \\ \sin(\theta) & \cos(\theta) & c\sin(\theta) + f\cos(\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

which is exactly what we expect.

Shear

Consider the following transformation:

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's suppose that $\theta = 36.9^0$ so cos(θ) = 0.8 and sin(θ) = 0.6. Consider what happens to the point (*a, a*) It becomes (0.8 *a* + 0.6 *a* , 0.6 *a* + 0.8 *a*) = (1.4 *a*, 1.4 *a*). In other words, all points on the 45^0 diagonal remain on the diagonal, but are stretched by a factor (cos(θ) + sin(θ))

Points on the X axis (*a*, 0) transform to (*a* cos(θ) , *a* sin(θ)) - ie they are rotated by an angle θ.

Points on the Y axis (0, *a*) transform to (*a* sin(θ) , *a* cos(θ)) - ie they are rotated by an angle θ but in a clockwise direction. The net effect is a kind of scissor action.

It is fairly obvious that any linear transformation can be undone by a translation (to restore the origin) followed by a rotation (to restore the 45^0 diagonal) followed by a shear (to make the axes orthogonal) followed by a scale (to restore the sizes of the axes)