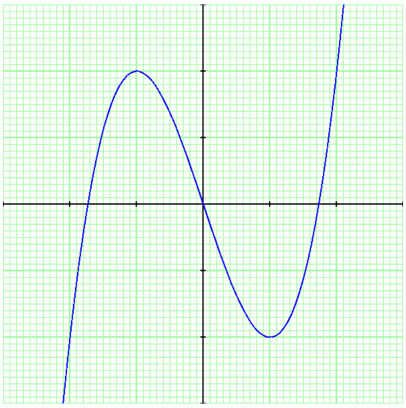
Maxima and Minima in 2 Dimensions

Stationary Points in 1 dimension

In one dimension, in order to determine whether a stationary point is a maximum, a minimum or an inflexion all we have to do is calculate the second differential. If it is positive, we have a minimum, negative we have a maximum and zero for an inflexion. e.g. consider the equation

$$y = x^3 - 3x \tag{1}$$

which looks like this:



Differentiating this we get

$$\frac{dy}{dx} = 3x^2 - 3 \tag{2}$$

which is zero at the points $x = 1$ and $x = -1$.

Differentiating again:

$$\frac{d^2y}{dx^2} = 6x \tag{3}$$

which equals -6 at $x = -1$ (a maximum) and +6 at $x = 1$ (a minimum)

So the rules which govern stationary points in 2 dimensions are simple: stationary points occur when the first differential is zero and the type of stationary point is determined by the value of the second differential: -1 = maximum, 0 = inflexion, +1 = minimum

Stationary points in 2 dimensions

Stationary points are such that the gradient in both the X and Y directions is zero. i.e.:

$$\frac{\partial V}{\partial x} = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = 0 \tag{4}$$

but how do we determine if such a point is a maximum, a minimum or something else like a saddle point? You can even have points which are part maximum and part inflexion.

First consider the equation:

$$V = x^2 + y^2 \tag{5}$$

This is a bowl-shaped surface with a minimum at the origin. We have:

$$\frac{\partial V}{\partial x} = 2x \quad \text{and} \quad \frac{\partial V}{\partial y} = 2y \tag{6}$$

both of which are zero at the origin.

Now there are 4 second order partial differentials: $\frac{\partial^2 V}{\partial x^2}$ $\frac{\partial^2 V}{\partial y^2}$ $\frac{\partial^2 V}{\partial x \partial y}$ and $\frac{\partial^2 V}{\partial y \partial x}$ (the last two being equal, of course).

These have the following values

$$\frac{\partial^2 V}{\partial x^2} = 2 \tag{7}$$

$$\frac{\partial^2 V}{\partial y^2} = 2 \tag{8}$$

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x} = 0 \tag{9}$$

The fact that both the first two are positive seems to indicate that the point must be a minimum but this is not always the case. Consider the equation:

$$V = x^2 + y^2 + 2xy \tag{10}$$

As before, both of the first partial differentials are zero at the origin so it appears to have a minimum there but this is not the case. If you consider the points where $x = -y$, (ie along a diagonal line to the axes) you will appreciate that they are *all* zero. The point (0,0) cannot therefore be a minimum. In fact the surface is not a bowl at all but a flat sheet curled into a parabola, touching the $z = 0$ plane along the line $x + y = 0$. At the origin, the surface is absolutely level and sections along the X and Y axes both show a minimum there but this is not a true minimum because for this to be the case, sections in *all directions* must show minima.

Indeed, if we consider the equation:

$$V = x^2 + y^2 + 3xy \tag{11}$$

then putting $x = -y$, we find that V is negative everywhere (except at the origin of course). This means that the origin is actually a saddle point

In general, if

$$V = x^2 + y^2 + axy \tag{12}$$

then

$$\partial^2 V_{xx} = \frac{\partial^2 V}{\partial x^2} = 2 \tag{13}$$

$$\partial^2 V_{yy} = \frac{\partial^2 V}{\partial y^2} = 2 \tag{14}$$

$$\partial^2 V_{xy} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x} = a^2 \tag{15}$$

In order to determine what kind of stationary point this is we must calculate the quantities:

$$T = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \partial^2 V_{xx} + \partial^2 V_{yy} \tag{16}$$

and

$$H = \frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \frac{\partial^2 V}{\partial x \partial y}\frac{\partial^2 V}{\partial y \partial x} = \partial^2 V_{xx}\,\partial^2 V_{yy} - (\partial^2 V_{xy})^2 \tag{17}$$

We now have multiple possibilities to consider:

If H is negative then the surface has negative curvature like a saddle regardless of the value of T.

If $H = 0$ then the surface at the point in question is *flat* (in the sense that the surface of a cylinder is flat) and the direction of curvature is given by T (negative for a minimum and positive for a maximum as usual).

If H is positive, then the surface has positive curvature like the surface of a ball and whether it is a maximum or minimum is again determined by T.

Applying these rules to equation (12) we have $H = 4 - a^2$ and $T = 4$ which tells us that, provided $a < 2$, we get a true minimum but if $a>2$ we get a saddle.

Lets try another equation:

$$V = x^2 - y^2 \tag{18}$$

$T = 0$ and $H = -4$ giving us a saddle at the origin.

What about:

$$V = x^2 + y^3 \tag{19}$$

The cubic curve has an inflexion at the origin so what sort of surface are we going to find there?

$\partial^2 V_{xx}=2$, $\partial^2 V_{yy}=0$ and $\partial^2 V_{xy}=0$ so $H = 0$ and $T = 2$. This tells us that the surface is flat with an upward curl along the X axis. It will look a bit like a chair.