Partitions

Definitions

Any integer n can be expressed as the sum of integers in $P(n)$ different ways.

Let $P_m(n)$ be the number of ways the number n can be written using integers no higher than m.

Obviously $P_1(n) = 1$

Now 6 can be expressed in 11 different ways:

6
5 + 1
4 + 2
4 + 1 + 1
3 + 3
3 + 2 + 1
3 + 1 + 1 + 1
2 + 2 + 2
2 + 2 + 1 + 1
2 + 1 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1 + 1

In the list, the partitions have been written in descending order (ie all the digits in any line are smaller than the largest digit in the line above) From this it is clear that, for example, $P_2(6) = 4$ and that $P_2(6) = P_1(5) + P_2(4)$ Or, to put it into words: The number of ways of expressing 6 as the sum of integers no higher than 2 is equal to the number of ways of expressing 5 in integers no higher than 1 plus the number of ways of expressing 4 in integers no higher than 2.

In general we have $$P_m(n) = \sum_{i=1}^{m} P_i(n-i)$$

and $$P(n) = P_n(n) = \sum_{i=1}^{n} P_i(n-i)$$

Now $P_2(4) = P_1(3) + P_2(2)$

and $P_2(2) = P_1(1) + P_2(0)$

from which we see that, in order to get the right result $P_2(0)$ must be equal to 1. This does in fact make sense because there is only one way of making zero out of numbers no higher than 2 – namely, use no numbers at all!.

If we use our formula to evaluate $P_2(0)$ we get

and $P_2(0) = P_1(0) + P_2(-1)$

Since $P_2(-1)$ is obviously zero, (you can't express -1 as the sum of any integers) we can write any expression of the form $P_m(0)$ as $P_1(0)$

Putting all these results together we see that $P_2(6) = P_1(5) + \{P_1(3) + \{P_1(1) + P_2(0)\}\} = 4$

Here is a table of values of $P_m(n)$. Bear in mind that $P_m(0) = 1$ and that $P_1(n) = 1$.

n / m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15	15	15	15	15	15	15	15	15	15	15
8	1	5	10	15	18	20	21	22	22	22	22	22	22	22	22	22	22	22	22	22
9	1	5	12	18	23	26	28	29	30	30	30	30	30	30	30	30	30	30	30	30
10	1	6	14	23	30	35	38	40	41	42	42	42	42	42	42	42	42	42	42	42
11	1	6	16	27	37	44	49	52	54	55	56	56	56	56	56	56	56	56	56	56
12	1	7	19	34	47	58	65	70	73	75	76	77	77	77	77	77	77	77	77	77
13	1	7	21	39	57	71	82	89	94	97	99	100	101	101	101	101	101	101	101	101
14	1	8	24	47	70	90	105	116	123	128	131	133	134	135	135	135	135	135	135	135
15	1	8	27	54	84	110	131	146	157	164	169	172	174	175	176	176	176	176	176	176
16	1	9	30	64	101	136	164	186	201	212	219	224	227	229	230	231	231	231	231	231
17	1	9	33	72	119	163	201	230	252	267	278	285	290	293	295	296	297	297	297	297
18	1	10	37	84	141	199	248	288	318	340	355	366	373	378	381	383	384	385	385	385
19	1	10	40	94	164	235	300	352	393	423	445	460	471	478	483	486	488	489	490	490
20	1	11	44	108	192	282	364	434	488	530	560	582	597	608	615	620	623	625	626	627

The first thing to notice is that the partition numbers $P(n)$ form the diagonal.

Secondly, all the values of $P_m(n)$ when $m > n$ (ie boxes to the right of the diagonal) are equal to $P(n)$. (You can't make any more partitions by using numbers greater than n)

Now, taking as an example $P_7(11) = 49$. We know that $P_7(11) = P_1(10) + ... + P_7(10)$ (These are shaded in red)

We also know that the next box to the right, $P_8(11) = P_1(10) + ... + P_8(10)$.

It follows that $P_8(11) = P_7(11) + P_8(3)$ (These are coloured in magenta)

In general $P_m(n) = \mathrm{P}_{m-1}(n) + P_m(n - m)$ (for $n <= m$) and this is how the table was constructed.