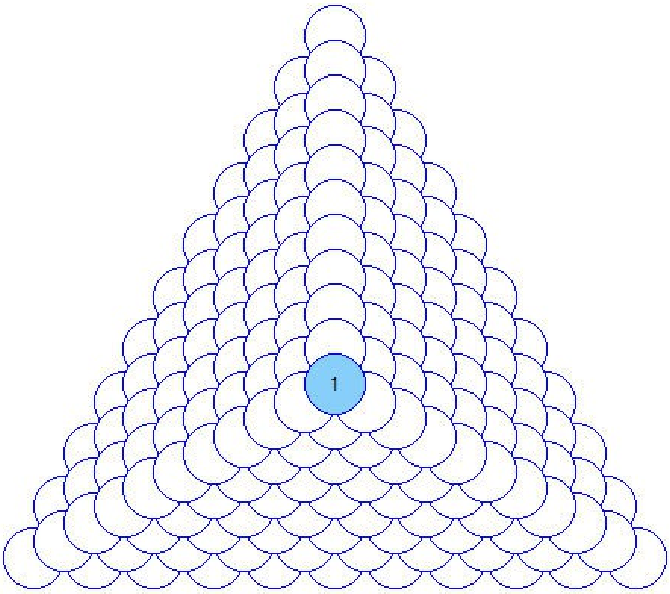
Pascal's Pyramids

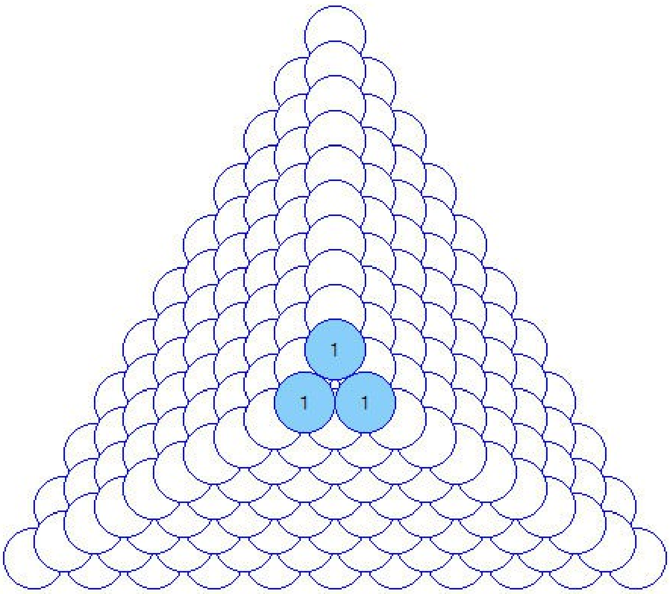
The Trinomial Pyramid

Pascal's triangle is well known. Each number in the triangle is the sum of the two in the line immediately above it. But what if we expand each line into a plane and require that each number is the sum of the three in the plane above? (We shall assume that the numbers are arranged like cannon balls stacked in a pyramid, each number having three close neighbours in the plane above; six in its own plane and three below: technically, this arrangement is known as a face-centred cubic lattice.)

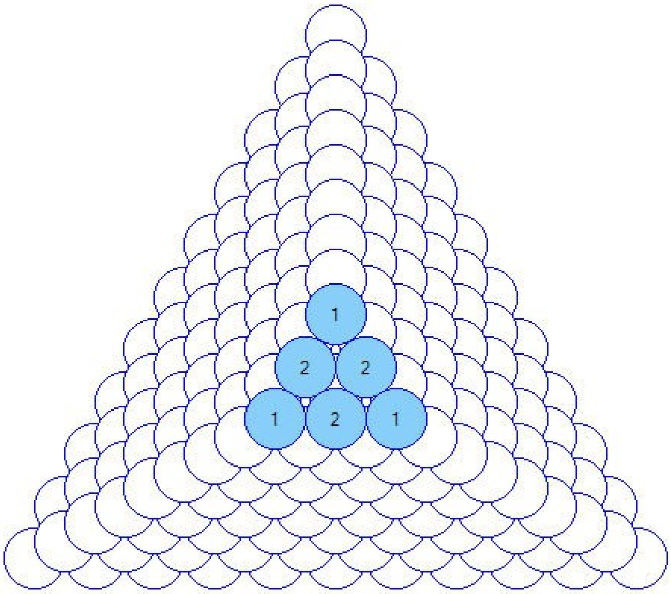
The first few planes will look like this



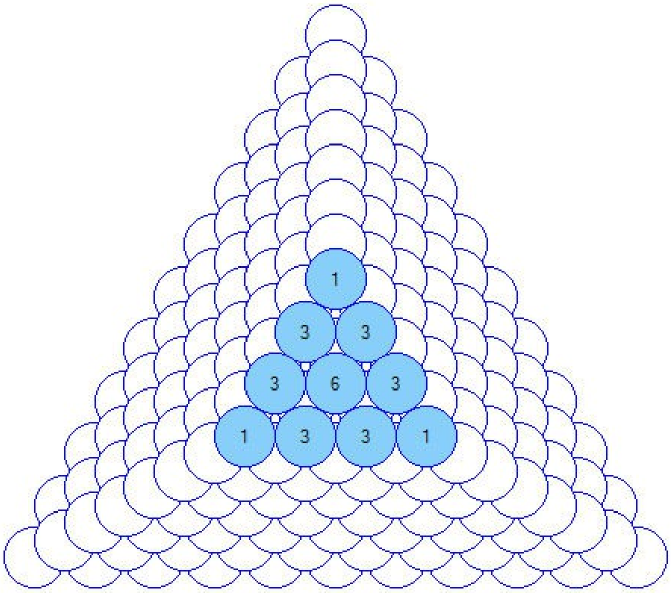
Plane 0 Total 1



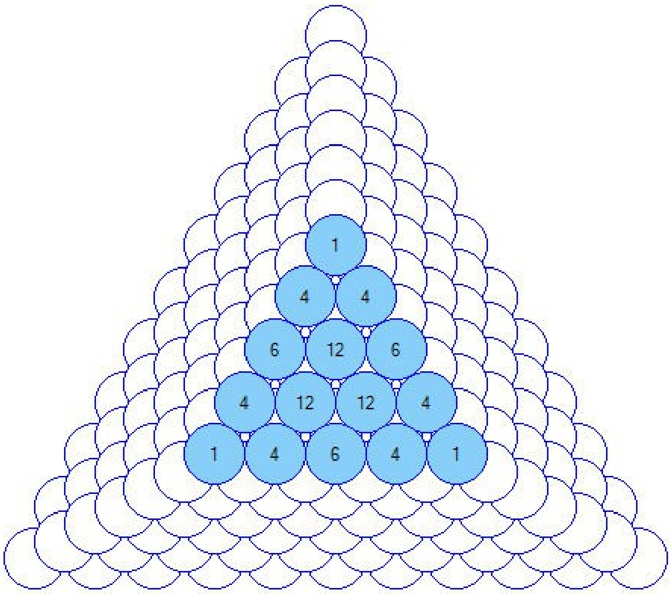
Plane 1 Total 3



Plane 2 Total 9



Plane 3 Total 27



Plane 4 Total 81

etc.

It will be noted that the sum of all the numbers in the plane is a power of 3.

It is also obvious that the sides of the pyramid are just Pascal's triangle.

Now it is quite difficult to visualise how the numbers are stacked on top of one another but a glance at the first picture above shows that the pyramid has the form of the corner of a cube. The position of each number is therefore uniquely specified by 3 coordinates *u*, *v* and *w* and the 'plane' in which the number lies is simply $u + v + w$.

Now according to the definition of the pyramid, the number at each coordinate is the sum of the three numbers immediately above it. It follows that each number is simply the total number of different ways in which you can travel from the origin to the coordinate in question. This is because if there are I*a* ways of reaching a certain number, *b* ways of reaching another and *c* ways of reaching a third, all arranged in a triangle in one of the planes, there will be $a + b + c$ ways of reaching the number directly beneath in the plane below.

This enables us to write down a general formula for the number at coordinate (*u*, *v*, *w*).

For example: in order to reach (3,5,2) in plane 10 you must make 3 X moves, 5 Y moves and 2 Z moves in any order you like. This is exactly equivalent to placing 10 differently coloured balls into three bags with 3 balls in bag X, 5 in bag Y and 2 in bag Z. In how many ways can this be done?

Well, there are 10! ways of placing 10 balls in order but since there are 3! ways in which the 3 balls in the first bag can go, 5! in the second and 2! in the third, the total number of different ways of reaching (3,5,2) is

$$\frac{10!}{3! \times 5! \times 2!} = 2520$$

and of reaching (*u*, *v*, *w*)

$$\frac{(u+v+w)!}{u! \times v! \times w!}$$

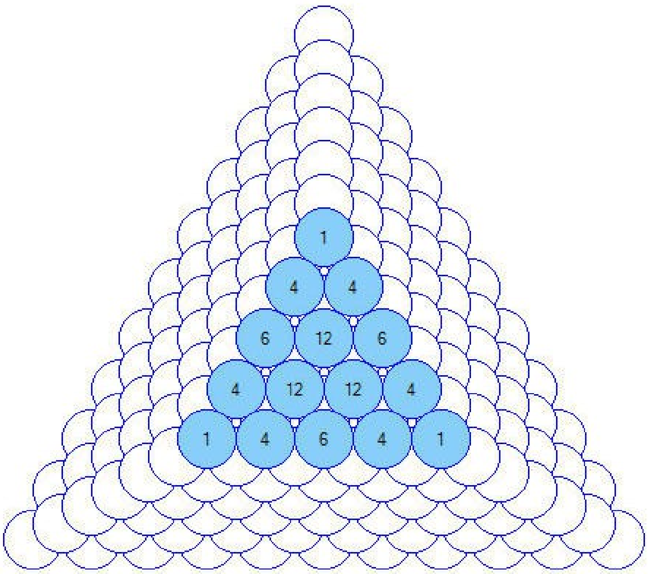
The Trinomial Theorem

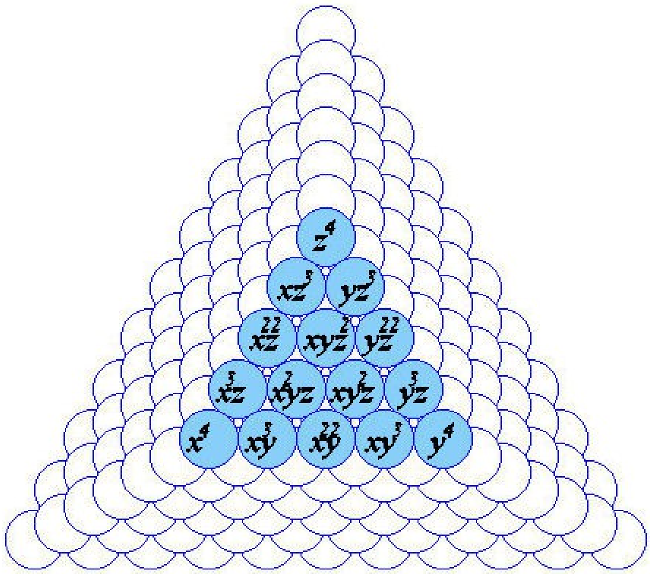
What are the coefficients of $(x + y + z)^n$?

Taking $n = 4$ as an example we get

$$x^4 + y^4 + z^4 + 4x^3 y + 4y^3 z + 4x z^3 + 4x^3 z + 4x y^3 + 4yz^3 +$$
$$6x^2 y^2 + 6y^2 z^2 + 6x^2 z^2 + 12x^2 y z + 12x y^2 z + 12xyz^2$$

and the relation with plane 4 is clear. If we label the three corner numbers x^4 $y^4$and z^4, we can allocate expressions to all the other cells like this:





We can also see why the coefficient of each term is the sum of the three coefficients above it.

When you multiply a polynomial expression by $(x + y + z)$, the coefficient of a certain term $x^p y^q z^r$ is the sum of three previous coefficients: $x^{p-1} y^q z^r \times$ x, $x^p y^{q-1} z^r \times$ y, $x^p y^q z^{r-1} \times$ z.

Links with Pascal's Triangle

If we look at the pyramid from the side and take it slice by slice, this is what we find:

1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
1	3	6	10	15	21	28	36	45	55
1	4	10	20	35	56	84	120	165	220
1	5	15	35	70	126	210	330	495	715
1	6	21	56	126	252	462	792	1287	2002
1	7	28	84	210	462	924	1716	3003	5005
1	8	36	120	330	792	1716	3432	6435	11440
1	9	45	165	495	1287	3003	6435	12870	24310
1	10	55	220	715	2002	5005	11440	24310	48620

Slice 0

It is obviously just Pascal's original triangle. The next slice looks like this:

1	2	3	4	5	6	7	8	9	10
2	6	12	20	30	42	56	72	90	110
3	12	30	60	105	168	252	360	495	660
4	20	60	140	280	504	840	1320	1980	2860
5	30	105	280	630	1260	2310	3960	6435	10010
6	42	168	504	1260	2772	5544	10296	18018	30030
7	56	252	840	2310	5544	12012	24024	45045	80080
8	72	360	1320	3960	10296	24024	51480	102960	194480
9	90	495	1980	6435	18018	45045	102960	218790	437580
10	110	660	2860	10010	30030	80080	194480	437580	923780

Slice 1

The first row is, of course, the second row of the original triangle and every diagonal is simply the triangle number multiplied by the number at each end.

Multidimensional Pyramids

All the ideas discussed above are easily extended to four or more dimensions.