Pascal's Triangle and the Summation of Polynomials

Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Pascal's triangle is constructed in the following way. The extremities of each line are always unity. The interior numbers are the sum of the two numbers in the row above.

The formula for the pth number of the nth row (counting the first number of each row and column as zero) is

$$\frac{n(n-1)(n-2)...(n-p+1)}{1.2.3... p} = \frac{n!}{(n-p)!. p!} = C_p^n$$

For example, the 2nd number in the 5th row (highlighted in red) is equal to

$$\frac{5.4}{1.2} = \frac{1.2.3.4.5}{1.2.3 \;\; . \;\; 1.2} = 10$$

The reason why the coefficients are related to the combinations is as follows. Consider the number of different routes whereby you can get from the first number to the highlighted one. Each time you move down a row you have a choice of moving either left or right. In order to get to the highlighted number (10) you must make exactly 2 right moves in a total of 5 – ie C_2^5

It follows from this that

$$C_{p-1}^{n-1} + C_p^{n-1} = C_p^n$$

and while this is not obvious from the above expressions, it is true because the only way to get to the highlighted 10 is through either the 4 or the 6 above it. The total number of ways of getting to the 10 must therefore be 4 + 6.

Another interesting property of the triangle is that the sum of all the coefficients in the n^{th} row is 2^n. This is true because the total number of different routes to any number in the n^{th} row is simply the number of different binary numbers with n digits.

A better way of constructing the triangle is like this

u/v	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	3	4	5	6
2	1	3	6	10	15	21
3	1	4	10	20	35	56
4	1	5	15	35	70	126
5	1	6	21	56	126	252

The red number (10) is at the position $u = 3$: $v = 2$ and the relation between u and v and n and p is that $n = u + v$ and $p = v$; hence the coefficient $P(u,v)$ is equal to

$$P(u,v) = \frac{(u+v)!}{u!.v!}$$

In this form, the symmetry of the table is shown clearly and the property that each number is the sum of the ones to the left and above is easily proved as follows:

$$\begin{aligned} & \frac{(u-1+v)!}{u!-1.v!} + \frac{(u+v-1)!}{u!.v!-1} \\ &= \frac{(u+v-1)!.u}{u!.v!} + \frac{(u+v-1)!.v}{u!.v!} \\ &= \frac{(u+v-1)!}{u!.v!}.(u+v) \\ &= \frac{(u+v)!}{u!.v!} \end{aligned}$$

Summing the integers

Consider the numbers in column $v = 1$. Each number is the sum of the number above it plus the number to the left – which is always 1. Each number is therefore 1 more than the one above and the sequence is simply the sequence of natural numbers 1, 2, 3, 4 etc.

In general the coefficient $P(u, 1)$ is equal to the sum of a list of one's from 0 to u. ie

$$P(u,1) = \sum_0^u 1 = u + 1$$

Of more interest are the numbers in column $v = 2$. It can easily be seen that these numbers are the sums of the integers. (6 is the sum of the numbers 1 to 3; add in a 4 and you have got 10, the sum of the integers 1 to 4). We can therefore say that

$$P(u,2) = \sum_1^{u+1} i = \frac{(u+2)!}{(u)!.(2)!} = \frac{(u+1).(u+2)}{2}$$

or more prosaically, putting $u = n - 1$,

$$P(u,2) = \sum_1^n i = \frac{n.(n+1)}{2}$$

A fact well known to the six-year old Gauss!

Summing the squares of the integers

In a similar way, the column $v = 3$ contains a list of the sums of the numbers in column $v = 2$

$$P(u,3) = \sum\sum i = \sum \frac{i.(i+1)}{2} = \frac{1}{2}\sum(i^2 + i) = \frac{1}{2}(\sum i^2 + \sum i)$$

(Note that all the sums are from 1 to n.)

Rearranging and putting in the formula for P(u,3) with $u = n - 1$, we find that

$$\sum i^2 = 2.\frac{(n-1+3)!}{(n-1)!.(3)!} - \frac{n.(n+1)}{2}$$
$$= \frac{n.(n+1).(n+2)}{3} - \frac{n.(n+1)}{2}$$

hence

$$\boxed{\sum i^2 = \frac{n.(n+1).(2n+1)}{6}}$$

Summing the Cubes of the integers

We now appear to have a general method for summing the p^{th} power of the integers because every column contains a running total of the columns which precede it and each column contains a term in the next higher power. The algebra gets quite complicated but a surprise awaits us when we sum the cubes of the integers. Watch!

$$P(u,4) = \sum P(u,3)$$
$$= \sum \frac{(u+3)!}{u!.3!}$$
$$= \sum \frac{(u+1).(u+2).(u+3)}{6}$$
$$= \sum \frac{n.(n+1).(n+2)}{6}$$
$$= \frac{1}{6}\sum(n^3 + 3n^2 + 2n)$$

Hence

$$\sum i^3 = 6.\mathrm{P}(n-1,4) - 3\sum n^2 - 2\sum n$$
$$= 6\frac{(n+3)!}{(n-1)!.(4)!} - 3\frac{n.(n+1).(2n+1)}{6} - 2\frac{n.(n+1)}{2}$$
$$= 6\frac{n.(n+1).(n+2).(n+3)}{1.2.3.4} - 3\frac{n.(n+1).(2n+1)}{6} - 2\frac{n.(n+1)}{2}$$
$$= \frac{n^2.(n+1)^2}{4}$$

or

$$\boxed{\sum i^3 = \left(\frac{n.(n+1)}{2}\right)^2}$$

This is a remarkable result. The expression $\frac{n.(n+1)}{2}$ is simply the sum of all the integers up to

n . We can therefore write this result as

$$\sum i^3 = \left(\sum i\right)^2$$

eg $1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100 = (1 + 2 + 3 + 4)^2$

For a graphical proof of this, see the end of this article.

Summing the fourth powers of the integers

$$P(u,5) = \sum P(u,4)$$
$$= \sum \frac{(u+4)!}{u!.\,4!}$$
$$= \sum \frac{(u+1).(u+2).(u+3).(u+4)}{24}$$
$$= \sum \frac{n.(n+1).(n+2).(n+3)}{24}$$
$$= \frac{1}{24}\sum (n^4 + 6n^3 + 11n^2 + 6n)$$

Hence

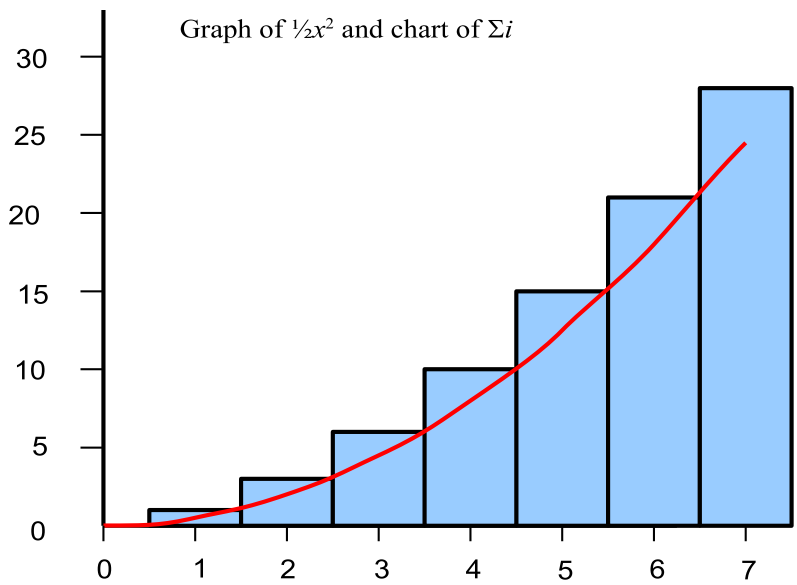
$$\sum i^4 = 24.\mathrm{P}(n-1{,}5) - 6\sum n^3 - 11\sum n^2 - 6\sum n$$

I will spare you the messy details but the answer is:

$$\boxed{\sum i^4 = \frac{n.(n+1).(2n+1).(3n^2+3n-1)}{30}}$$

The relation between summation and integration

There is obviously a similarity between $\sum_1^n i^p$ and $\int_0^a x^p\,dx$ which we can illustrate on a graph



It is clear that $\sum_1^n i^p$ is always slightly larger than $\int_0^a x^p\,dx$

Here is a table of results comparing summation and integration. (The expressions for the sums have been normalized so that the denominator is equal to the power)

Power	Sum	Integral
i^0	n	a
i^1	$\frac{n.(n+1)}{2}$	$\frac{a^2}{2}$
i^2	$\frac{n.(n+1).(n+1/2)}{3}$	$\frac{a^3}{3}$
i^3	$\frac{n.n.(n+1).(n+1)}{4}$	$\frac{a^4}{4}$
i^4	$\frac{n.(n+1).(n+1/2).(n^2+n-1/3)}{5}$	$\frac{a^5}{5}$

Summing higher powers

While the integrals show an obvious pattern, it is not clear to me how one could guess the formula for $\sum i^5$ though it seems probable that it contains the factors $n.(n+1)$ and when expressed in a form like the above, it will have a denominator of 6.

Here is a table of values for $\sum i^5$ with some of the prime factors

1	2	3	4	5	6
1	32	243	1024	3125	7776
1	33 = 3 x 11	276 = 12 x 23	1300 = 100 x 13	4425 = 75 x 59	12201 = 3 x 7 x 7 x 83

If we make a plausible guess that the expression has the form

$$\frac{n.(n+1).(n+p).(n+q).(an^2+bn+c)}{6a}$$

and that the (an^2+bn+c) term is responsible for the factors of 11, 23 and 59 when $n = 2$, 3 and 5 respectively, then it seems probable that this term is in fact $(2n^2+2n-1)$ – it being highly unlikely that a random set of three simultaneous equations in three variables could have such a simple solution! This is amply confirmed by the presence of the factor 13 and 83 when n = 4 and n = 6. It is easy to see that the complete expression must be

$$\sum i^5 = \frac{n.n.(n+1).(n+1)(2n^2+2n-1)}{12}$$

One obvious pattern which emerges is that all sums (of powers greater than 1) have factors n and $(n+1)$ and that odd powers have factors n^2 and $(n+1)^2$. In addition the even powers appear to have the factor $(2n+1)$ as well

In the case of $\sum i^6$ the denominator must have a factor of 7. This puts severe constraints on the possible factors in the numerator because at least one of them must be divisible by 7 for every value of n. One possible expression for the numerator is

$$n.(n+1).(n+2).(n+3).(n+4).(n+5).(n+6)$$

This is obviously not correct, but we can mess about with it in a variety of ways without destroying the property that it is always divisible by 7. We can, for example, multiply any of the terms by any number and we can add (or subtract) any multiple of 7. eg take the term $(n + 4)$. If this is divisible by 7 then so is $(2n + 8)$ and so is $(2n + 1)$ We can also multiply any of the terms together and then add or subtract multiples of 7. For example we could take the term $(2n + 1)$ and multiply it by $(n + 5)$ to get $(2n^2 + 11n + 5)$, subtract a few 7's and divide by 2 to get $(n^2 + 2n - 1)$ which will do the job of both the $(n + 4)$ and $(n + 5)$ terms. Obviously we can generate a huge number of possible terms in this way but it does rule out a lot of possibilities.

Here is a table of values for $\sum i^6$ with some of the prime factors

1	2	3	4	5	6
1	64	729	4096	15625	46656
1	65 (5 x 13)	794 (2 x 397)	4890 (30 x 163)	20515 (5 x 11 x 373)	67171

The presence of a large prime factor (397) when $n = 3$ is interesting and suggests that one of the factors may have an n^3 term or even an n^4 term. so a plausible expression would be

$$\sum i^6 = \frac{n.(n+1).(2n+1).(an^4 + ...)}{7z}$$

(A search on Google revealed that

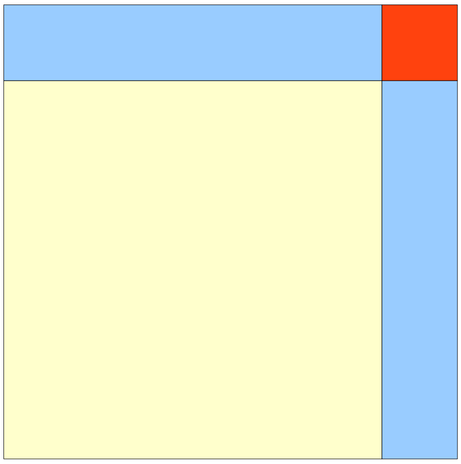
$$\sum i^6 = \frac{n.(n+1).(2n+1).(3n^4 + 6n^3 - 3n + 1)}{42}$$

Formulae for even higher powers follow no obvious pattern but the expected simpler factors are all there.)

An excellent account of the history of the sums of the powers of the integers is given in

http://mathdl.maa.org/mathDL/?pa=content&sa=viewDocument&nodeId=3284&bodyId=3531

Graphical proof of $\sum i^3 = \left(\sum i\right)^2$



Consider a square of side equal to 1 + 2 + 3 + ... + *n*. The area of this square can be calculated in two ways

$$\text{Area of large square} = (1 + 2 + 3 + \ldots + n)^2 = \left(\frac{n.(n + 1)}{2}\right)^2$$

Now let us divide the square by marking off two stripes of width *n*. The area of the large square is now equal to the area of the yellow smaller square plus the area of the red square plus the area of the two blue rectangles

$$\text{Area of large square} = \text{Area of yellow square} + n^2 + 2.\text{n}.(1 + 2 + 3 + \ldots + (n - 1))$$

$$= \text{Area of yellow square} + n^2 + 2n\left(\frac{(n-1).n}{2}\right)$$

$$= \text{Area of yellow square} + n^3$$

But the yellow square can also be divided in a similar way into a smaller square plus an L-shaped piece of area $(n - 1)^3$. In fact the whole square can be divided into a whole series of L-shaped pieces whose total are is $1^3 + 2^3 + 3^3 + \ldots + n^3$

It follows that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n.(n + 1)}{2}\right)^2 = \left(\sum_{i=1}^{n} i\right)^2$$

A general method for deriving a polynomial formula for Σ i ^{k}

(This article was written before the above and approaches the same problem from a different direction)

$$\sum_{i=1}^{n} i^k \;=\; 1^k+2^k+3^k+4^k+\ldots n^k$$

This process is not dissimilar to the process of integrating x^k and, by analogy we might expect to find that there exists a polynomial function for Σi^k which is of the order of $k + 1$. Hence let us suppose that

$$\sum_{i=1}^{n} i^k \;=\; a_0+a_1 n+a_2 n^2+a_3 n^3+\ldots a_k n^k+a_{k+1} n^{k+1}$$

Now since when $n = 0$

$$\sum_{i=1}^{0} i^k \;=\; 0$$

it follows that $a_0 = 0$.

Now since

$$\sum_{i=1}^{n} i^k \;-\; \sum_{i=1}^{n-1} i^k \;=\; n^k$$

we require that

$$a_1 n+a_2 n^2+a_3 n^3+\ldots a_k n^k+a_{k+1} n^{k+1}-[a_1(n-1)+a_2(n-1)^2+a_3(n-1)^3+\ldots a_k(n-1)^k+a_{k+1}(n-1)^{k+1}]=n^k$$

$$a_1 n-a_1(n-1)+a_2 n^2-a_2(n-1)^2+a_3 n^3-a_3(n-1)^3+\ldots a_k n^k-a_k(n-1)^k+a_{k+1} n^{k+1}-a_{k+1}(n-1)^{k+1}=n^k$$

$$a_1(n-(n-1))+a_2(n^2-(n-1)^2)+a_3(n^3-(n-1)^3)+\ldots a_k(n^k-(n-1)^k)+a_{k+1}(n^{k+1}-(n-1)^{k+1})=n^k$$

for all values of n. What this means is that the coefficients of all the terms on each side of the equation must match.

Consider the term n^{k+1}. It is clear that this term cancels out and therefore provides us with no information. (It is also worth noting that the first term of every other expansion of $(n – 1)^x$ also cancels out.)

Now consider the term n^k. Here we have

$$a_{k+1}(k+1)n^k \;=\; n^k$$

Hence

$$a_{k+1} \;=\; \frac{1}{(k+1)}$$

Now consider the term n^{k-1}. Here we have

$$-a_k k\, n^{k-1} \;+\; a_{k+1}\frac{(k+1).k}{2} n^{k-1} \;=\; 0$$

Hence

$$a_k \;=\; a_{k+1}\frac{(k+1).k}{2k} \;=\; a_{k+1}\frac{k+1}{2} \;=\; \frac{1}{k+1}.\frac{k+1}{2}$$

$$a_k = \frac{1}{2}$$

Now consider the term n^{k-2}. Here we have

$$-a_{k-1}(k-1)n^{k-2} \ + \ a_k\frac{k.(k-1)}{2}n^{k-2} \ - \ a_{k+1}\frac{(k+1).k.(k-1)}{2.3}n^{k-2} \ = \ 0$$

Hence

$$a_{k-1}(k-1)n^{k-2} \ = \ a_k\frac{k.(k-1)}{2}n^{k-2} - a_{k+1}\frac{(k+1).k.(k-1)}{2.3}n^{k-2}$$

$$a_{k-1} \ = \ \frac{1}{2}.\frac{k}{2} - \frac{1}{k+1}\frac{(k+1).k}{2.3} = k(\frac{1}{4}-\frac{1}{6})$$

$$a_{k-1} \ = \ \frac{k}{12}$$

The method for working out the coefficients should now be clear. Working from the highest power back to the start, each equation allows us to work out the next coefficient down. Lets do a couple more.

Consider the term n^{k-3}. Here we have

$$a_{k-2}(k-2)n^{k-3} - a_{k-1}\frac{(k-1).(k-2)}{2}n^{k-3} + a_k\frac{k.(k-1).(k-2)}{2.3}n^{k-3} - a_{k+1}\frac{(k+1).k.(k-1).(k-2)}{2.3.4}n^{k-3} = 0$$

Hence

$$a_{k-2}(k-2)n^{k-3} = a_{k-1}\frac{(k-1).(k-2)}{2}n^{k-3} - a_k\frac{k.(k-1).(k-2)}{2.3}n^{k-3} + a_{k+1}\frac{(k+1).k.(k-1).(k-2)}{2.3.4}n^{k-3}$$

$$a_{k-2} \ = \ a_{k-1}\frac{(k-1)}{2} \ - \ a_k\frac{k.(k-1)}{2.3} \ + \ a_{k+1}\frac{(k+1).k.(k-1)}{2.3.4}$$

$$a_{k-2} \ = \ \frac{k}{12}.\frac{(k-1)}{2} - \frac{1}{2}\frac{k.(k-1)}{2.3} \ + \ \frac{1}{k+1}\frac{(k+1).k.(k-1)}{2.3.4} \ = \ k(k-1)\left(\frac{1}{24}-\frac{1}{12}+\frac{1}{24}\right)$$

$$a_{k-2} \ = \ 0$$

By now it should be possible to write down the formula for the next coefficient as follows:

$$a_{k-3} \ = \ a_{k-2}\frac{(k-2)}{2} \ - \ a_{k-1}\frac{(k-1).(k-2)}{2.3} \ + \ a_k\frac{k.(k-1)(k-2)}{2.3.4} \ - \ a_{k+1}\frac{(k+1).k.(k-1).(k-2)}{2.3.4.5}$$

$$a_{k-3} \ = \ 0 \ - \ \frac{k}{12}.\frac{(k-1).(k-2)}{2.3} \ + \ \frac{1}{2}.\frac{k.(k-1).(k-2)}{2.3.4} \ - \ \frac{1}{k+1}\frac{(k+1).k.(k-1).(k-2)}{2.3.4.5}$$

$$a_{k-3} \ = \ k.(k-1).(k-2)\left(-\frac{1}{72}+\frac{1}{48}-\frac{1}{120}\right)$$

$$a_{k-3} \ = \ -k.(k-1).(k-2)\left(\frac{1}{720}\right)$$

What we have found out so far can be summarised in a table.

k	a_{k+1}	a_k	a_{k-1}	a_{k-2}	a_{k-3}	polynomial	factored polynomial
0	1/1					n	n
1	1/2	1/2				$n^2/2 + n/2$	$n(n + 1)/2$
2	1/3	1/2	1/6			$n^3/3 + n^2/2 + n/6$	$n(n + 1)(2n + 1)/6$
3	1/4	1/2	1/4	0		$n^4/4 + n^3/2 + n^2/4$	$n^2(n + 1)^2/4$
4	1/5	1/2	1/3	0	–1/30	$n^5/5 + n^4/2 + n^3/3 - n/30$	$n(n + 1)(2n + 1)(3n^2 + 3n - 1)/30$

Summing a polynomial

The summation operator acts like a kind of integral. In particular, if you want to sum a polynomial expression, all you have to do is sum each term separately.

Let $$S = c_0 + c_1 i + c_2 i^2 + c_3 i^3 + \ldots$$

Then $$\sum_{i=1}^{n} S = \sum_{i=1}^{n} c_0 + \sum_{i=1}^{n} c_1 i + \sum_{i=1}^{n} c_2 i^2 + \sum_{i=1}^{n} c_3 i^3 + \ldots$$

so $$\sum_{i=1}^{n} S = c_0 n + c_1 \frac{n(n+1)}{2} + c_2 \frac{(n+1)(2n+1)}{6} + c_3 \frac{n^2(n+1)^2}{4} + \ldots$$

For example, what is the sum of the following series: 1.2 + 2.3 + 3.4 + .. $n(n + 1)$

$$\sum_{i=1}^{n} i(i+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

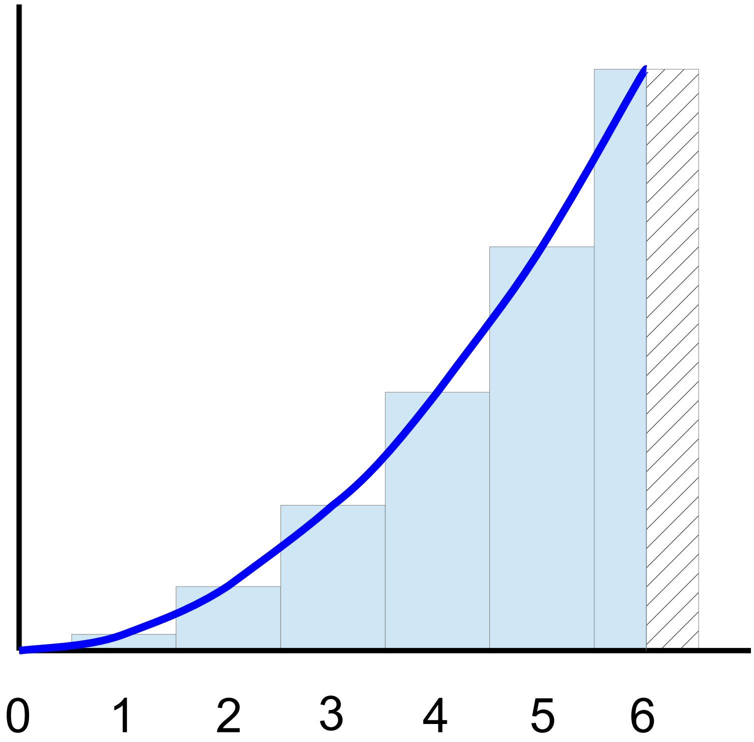
As another example, what is the sum of the following series: $1.2^2 + 2.3^2 + 3.4^2 + .. n(n + 1)^2$

$$\sum_{i=1}^{n} i(i+1)^2 = \sum_{i=1}^{n} i^3 + 2\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i(i+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12}$$

The relation between summation and integration

There is an obvious similarity between $\sum_{i=1}^{n} i^k$ and $\int_0^n x^k$ which can be illustrated as follows:



It is clear that the total area of the vertical bars (excluding the hatched area) is very close to the area under the curve. Now the hatched area has the value $n^k/2$ and is in fact always the second term in the expression for the sum. The integral is just the first term in the summation so the difference between the two areas is very small – being the sum of the third and subsequent terms only.

k	summation	integral
0	n	n
1	$n^2/2 + n/2$	$n^2/2$
2	$n^3/3 + n^2/2 + n/6$	$n^3/3$
3	$n^4/4 + n^3/2 + n^2/4$	$n^4/4$
4	$n^5/5 + n^4/2 + n^3/3 - n/30$	$n^5/5$

When n is large, the two expressions will become identical because the curve will become straighter and straighter.