Solid angles and Polyhedra

Just as an angle θ is defined as the ratio of the arc length to the radius of a circle, the solid angle subtended by an arbitrary area on the surface of a sphere is defined as the ratio of the area of the surface to the square of the radius. Since the surface area of a sphere is $4\pi R^2$, the solid angle of a sphere is 4π steradians.

Solid angle of a cone of semi-angle α

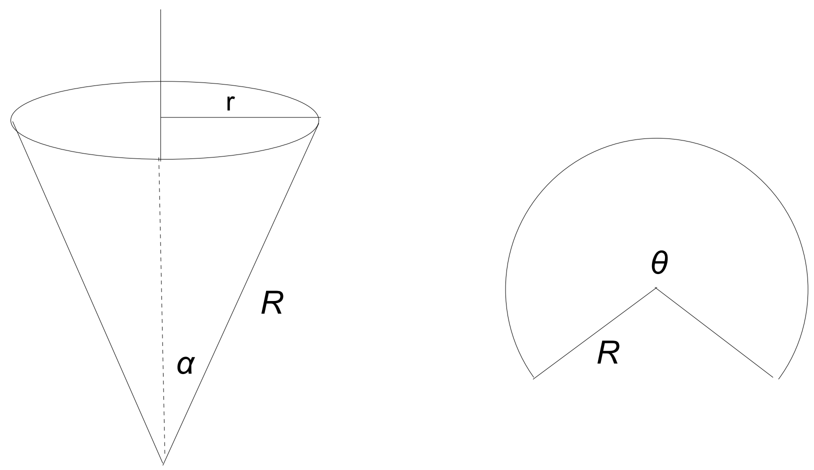
Area of thin ring subtending a semi-angle σ at the centre of a sphere $= 2\pi R \sin\sigma \,.\, R\delta\sigma$

Area of cap of semi-angle α $\quad A = \int_0^\alpha 2\pi R \sin\sigma \,.\, R\delta\sigma = 2\pi R^2(1 - \cos\alpha)$

hence
$$\boxed{A = 2\pi(1 - \cos\alpha)}$$

Solid angle of a paper cone

Consider a cone made out of a sector of a circle of radius R whose apex angle is θ.



The arc length of the edge of the cone is $R\theta$, hence $\quad r = R\theta/2\pi$

We need to know the cosine of α.

$$\cos\alpha = \frac{\sqrt{R^2 - r^2}}{R} = \sqrt{1 - \frac{r^2}{R^2}} = \sqrt{1 - \frac{\theta^2}{4\pi^2}}$$

So we can now deduce the solid angle of the cone:

$$A = 2\pi\left(1 - \sqrt{1 - \frac{\theta^2}{4\pi^2}}\right)$$

$$\boxed{A = 2\pi - \sqrt{4\pi^2 - \theta^2}}$$

Exterior angle of a paper cone

The solid angle of cone whose apex angle is 2π (i.e. a flat sheet of paper) is 2π.

If we want to consider solid angles enclosed by the corners of a polyhedron, it is more important to know the *exterior* solid angle of the corners i.e.:

$$A_{ext} = 2\pi - A = \sqrt{4\pi^2 - \theta^2}$$

Linton's conjecture

It was my belief that if you construct a corner of a polyhedron out of a paper cone, the same formula will apply. Lets see if it works for the corner of a cube.

We have $\theta = 3\pi/2$ and $A_{ext} = 2\pi \frac{\sqrt{7}}{4} = 2\pi \times 0.6614$

Since 8 cubic corners can be stacked round a single point, each would have an interior solid angle of $4\pi/8$ or $2\pi/4$ and an exterior solid angle of $2\pi \frac{\sqrt{3}}{4} = 2\pi \times 0.75$

This is obviously a bit different from the figure calculated above.

Now, the presence of the irrational number $\sqrt{7}$ indicates that the formula for the paper cone angle can never give an exact answer. This is because, if you actually construct a cone with apex angle $3\pi/2$ out of paper, it is obvious that when you deform it into the corner of a cube, you decrease the interior solid angle slightly (and increase the exterior angle).

The sum of the exterior angles

In the case of regular polygons, the sum of the exterior angles is always equal to 2π because in making a circle round the polygon you make one complete turn. Is there a similar theorem for solid angles?

It is a well known theorem of spherical trigonometry that the area of a spherical triangle (on a unit sphere) is equal to the sum of the interior angles of the triangle – π. (For a proof, see below.)

For example, the area of a triangle with three right angles is therefore $\pi/2$ (This is correct because 8 corners of a cube can be stacked round a single point.) The exterior solid angle is therefore

$$A_{ext} = 2\pi - \frac{\pi}{2} = 3\frac{\pi}{2}$$

So the sum of all 8 exterior solid angles = $8 \times 3\pi/2 = 12\pi$.

Could it be that the exterior solid angles of any polygon add up to 12π?

To work out the case of the tetrahedron, we need to know the corner angles of a spherical triangle whose vertices form a tetrahedron with the centre of the sphere.

Consider two planes which form one corner of this triangle and which include two of the triangles which make up the tetrahedron. It is easy to see that the angle we require (the corner angle) is simply the angle between the two planes. Some simple trig shows this to be $\cos^{-1}(1/3) = 70.5^0 = 0.392\pi$.

The area of the triangle (and hence the solid angle of the tetrahedron) is therefore $3\times 0.392\pi - \pi = 0.175\pi$. This means that the exterior solid angle of a tetrahedron is 1.824π and the total exterior solid angle is 7.3π. Our conjecture is clearly incorrect.

The vertex deficit theorem

If we add up all the angles of the faces which meet at a vertex and subtract this from 2π, we get what is called the vertex deficit δ. In a way this is more like the exterior angle of a polygon than the exterior solid angle, not least because *the sum of the vertex deficits of any polyhedron is equal to 4π.*

For example: the vertex deficit of a corner of a tetrahedron is $2\pi - 3 \times \pi/3 = \pi$. And since there are 4 vertices, the total deficit is 4π.

As another example, take the square drum which has 8 vertices, each of which comprises a square and three 60^0 triangles. The vertex deficit is therefore $2\pi - \pi/2 - 3 \times \pi/3 = \pi/2$ and the total deficit is once again 4π.

The statement about vertex deficits is exactly equivalent to Euler's theorem.

First we have:

total vertex deficit = 2π × number of vertices – sum of all the interior angles

Now for any polygon with N sides, the sum of the interior angles is $(N - 2)\pi$.

$$\text{total vertex deficit} = 2\pi V - \sum (N - 2)\pi = 4\pi$$

hence
$$2V - \sum N + \sum 2 = 4$$

where the sum is over all the faces.

Now since each edge is counted twice, we know that $\sum N = 2E$

and, of course, $\sum 2 = 2F$ hence

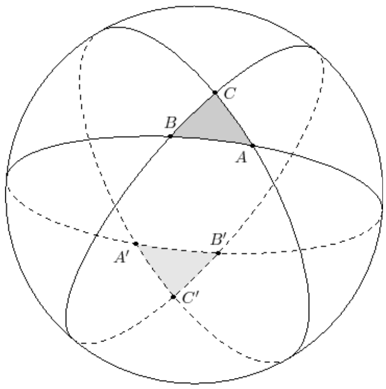
$$2V - 2E + 2F = 4$$
$$V + F = E + 2$$

We can now see the analogy between the two cases. Compare *the sum of the exterior angles of a polygon = 2π* with *the sum of the vertex deficits of any polyhedron = 4π.*

(I had hoped that the vertex deficit theorem would help to determine the angles of a polyhedron whose *FVE* statistics were known but this is not the case as it contains no new information over and above Euler's theorem. For example: consider the solid made by sticking the edges of 4 parallelograms. The solid has F=4, V=6 & E=8 but the angle deficit theorem does not help us to work out the angles of the parallelogram for the simple reason that any angle will do. Nor does it alert us to the fact that the parallelogram faces are bent!

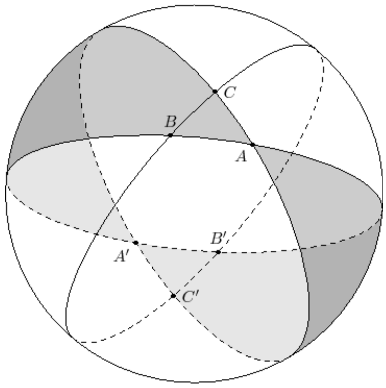
If we insist on the faces being flat, there is only one solution – the acute angle of the parallelogram must be zero.)

Area of a spherical triangle (proof)

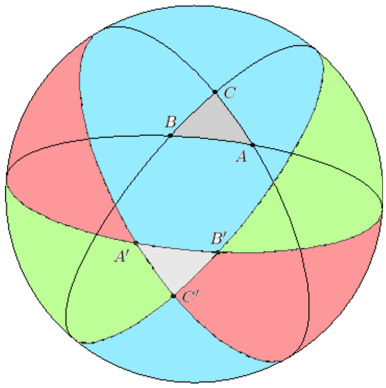


If you continue the great circles which define the triangle ABC right round the sphere, you divide the sphere into 8 areas of which one is a copy of the triangle whose area we wish to determine.. Now the area of the shaded area in the diagram below (defined by the angle *A*) is equal to

$4\pi \times \frac{A}{\pi} = 4A$. Similarly the areas of the other segments will be 4*B* and 4*C*. (We shall assume for simplicity that *A*, *B* and *C* are all acute angles.)



If we painted all three segments with reg, green and blue paint, we would cover the whole sphere but we would paint each of the opposed triangles three times (ie 4 extra triangles are painted)



This means that, if the area of the triangle is *S* then

$$4A + 4B + 4C = 4\pi + 4S$$

So $$S = A + B + C - \pi$$

It follows that the solid angle subtended at the centre of a sphere by a spherical triangle is also equal to $A + B + C - \pi$