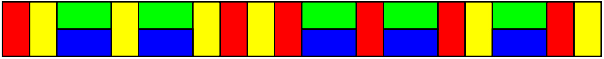
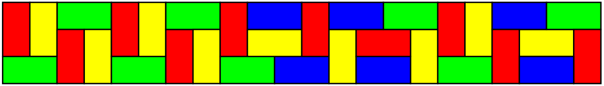


Tatami tilings: structure, enumeration and puzzles

Frank Ruskey (with Alejandro Erickson, Mark Schurch, Jennifer Woodcock)

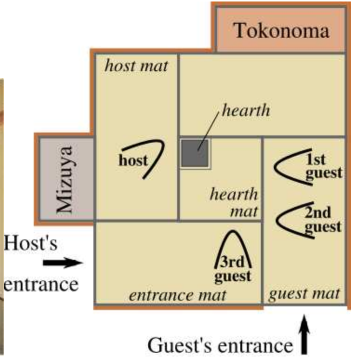


Workshop on Algorithmic Graph Theory, Nipissing, Ontario, May 2011

Lucky Tea House Floors

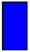
Tatami mats

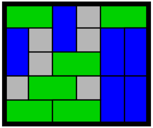
Traditional Japanese floor mats made of soft woven straw. They are either square or have a 1×2 aspect ratio.



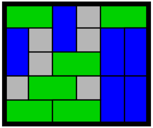
Certain floors, like tea houses, required that no four mats touch at any point.

Monomer-dimer tiling:

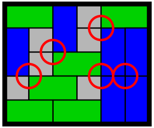
Tile a subset of the integer lattice with monomers (), and dimers ( and ).



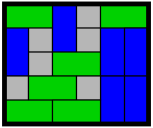
Monomer-dimer tiling:
Tile a subset of the integer lattice with monomers ( ), and dimers (and).



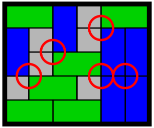
Tatami restriction: No tilings allowed with four tiles meeting at a point.



Monomer-dimer tiling:
Tile a subset of the integer lattice with monomers ( ), and dimers (and).

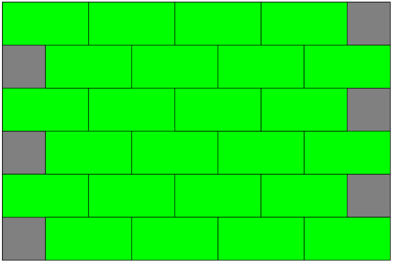


Tatami restriction: No tilings allowed with four tiles meeting at a point.

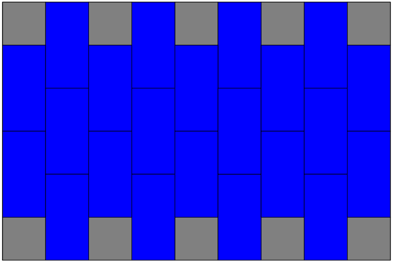


Graph theory interpretation: In a grid graph G, a matching M such that $G - M$ contains no 4-cycles.

The trivial tilings



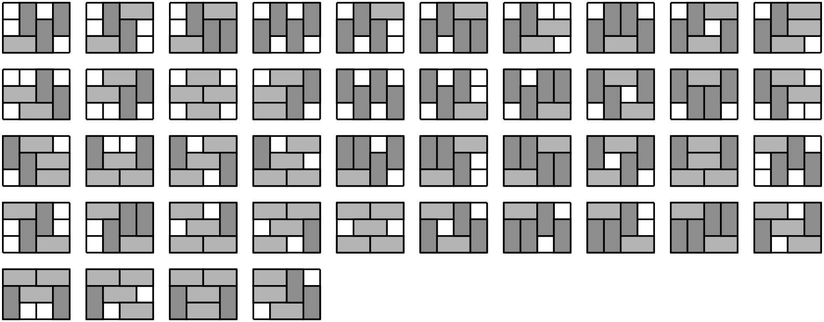
Horizontal “running bond”



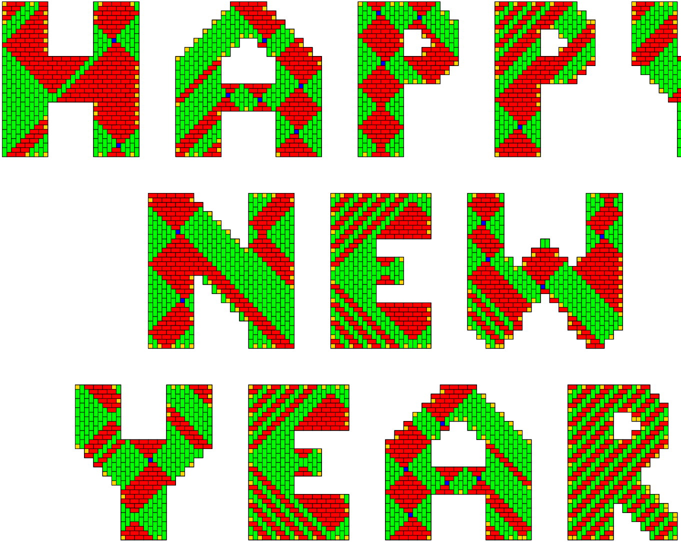
Vertical “running bond”

Small tilings

All 44 tatami tilings of the 3×4 grid.



Happy New Year message, 2010



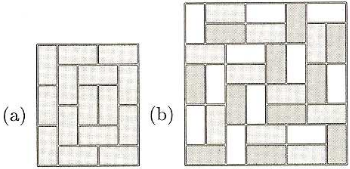
From Knuth volume 4A, Fascicle 1

First printing, March 2009, Exercise 7.1.4 #215.

215. [*21*] Japanese tatami mats are 1×2 rectangles that are traditionally used to cover rectangular floors in such a way that no four mats meet at any corner. For example, Fig. 29(a) shows a 6×5 pattern from the 1641 edition of Mitsuyoshi Yoshida's *Jinkōki*, a book first published in 1627.

Find all domino coverings of a chessboard that are also tatami tilings.

Fig. 29. Two nice examples:
(a) A 17th-century tatami tiling;
(b) a tricolored domino covering.



Solution

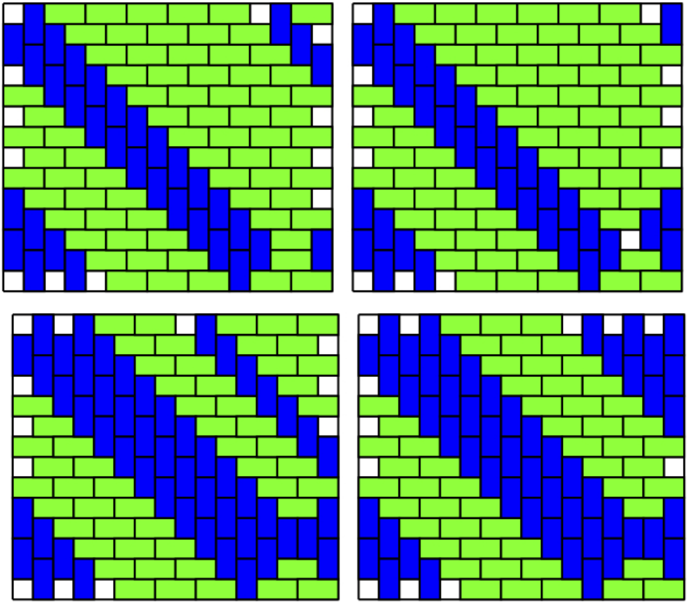
215. This time we add the constraints $\bigwedge_{j=1}^{49} S_{\geq 1}(Z_j)$, where Z_j is the set of four placements x_i that surround an internal corner point. (For example, $Z_1 = \{x_1, x_2, x_4, x_{16}\}$.) These constraints reduce the ZDD size to 66. There are just two solutions, one the transpose of the other, and they can readily be found by hand. [See Y. Kotani, *Puzzlers' Tribute* (A. K. Peters, 2002), 413–420.]

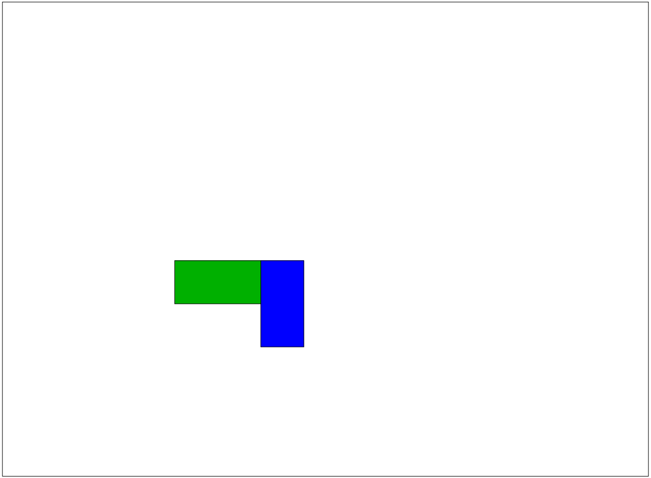
Conjecture: The generating function for the number of $m \times n$ tatami tilings, when $n \geq m - 2 \geq 0$ and m is even, is $(1+z)^2(z^{m-2}+z^m)/(1-z^{m-1}-z^{m+1})$.

Previous work on tatami tilings:

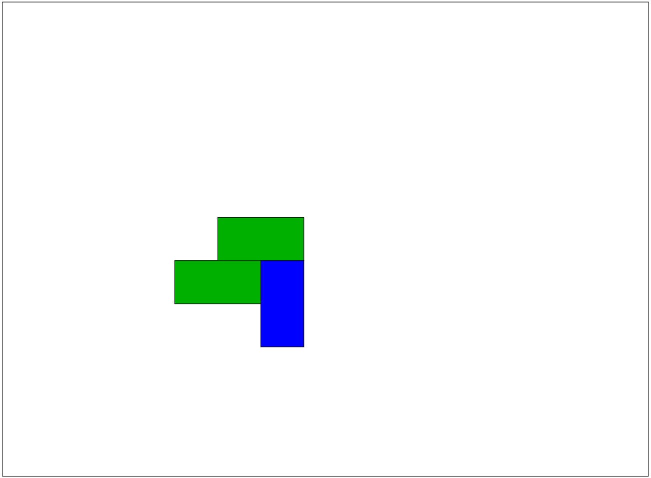
- **Kotani, 2002:** *Tatami Tilings*, in A Puzzler's Tribute: A Feast for the Mind.
- **Hickerson, 2002:** OEIS a068920 (and other OEIS entries) `http://www.research.att.com/~njas/sequences/`.
- **Knuth, 2009:** The Art of Computer Programming, volume 4, fascicle 1B.
- **R. and Woodcock, 2009:** Counting Fixed-Height Tatami Tilings, Electronic J. of Combinatorics, Paper R126 (2009) 20 pages.
- **Alhazov, Morita, and Iwamoto, 2010:** A note on tatami tilings, Mathematical Foundation of Algorithms and Computer Science, RIMS Kôkyûroku series, No. 1691, Research Institute for Mathematical Sciences, Kyoto, Japan, (2010), 1–7.
- **Erickson, R., Schurch, and Woodcock, 2010:** Auspicious Tatami Mat Arrangements, 16th COCOON Conference, LNCS 6196, pp. 288–297. Updated version to appear in the Electronic Journal of Combinatorics.

Larger tilings suggest structure

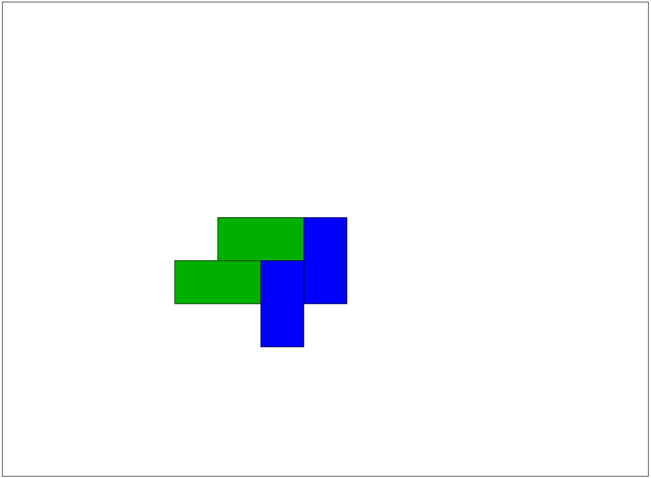




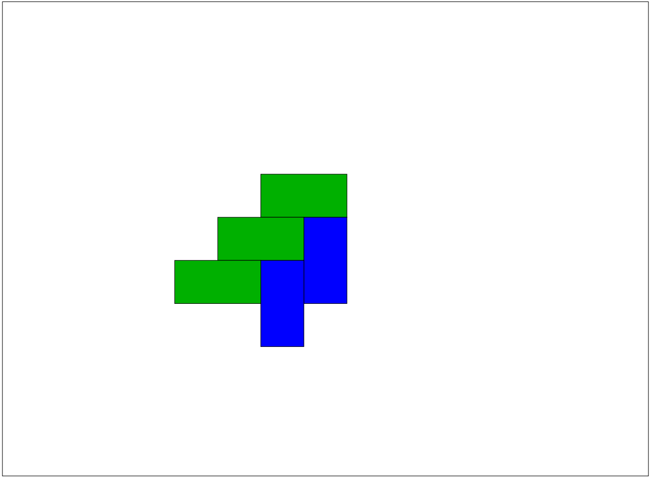
What are the consequences of this arrangement?



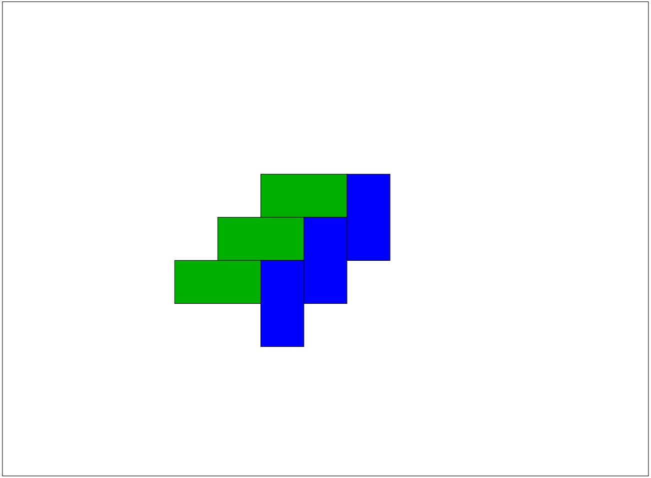
This placement is forced.



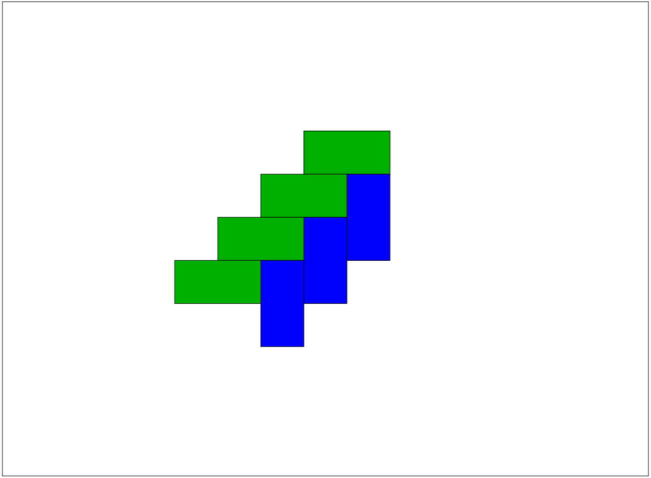
And this placement is also forced.



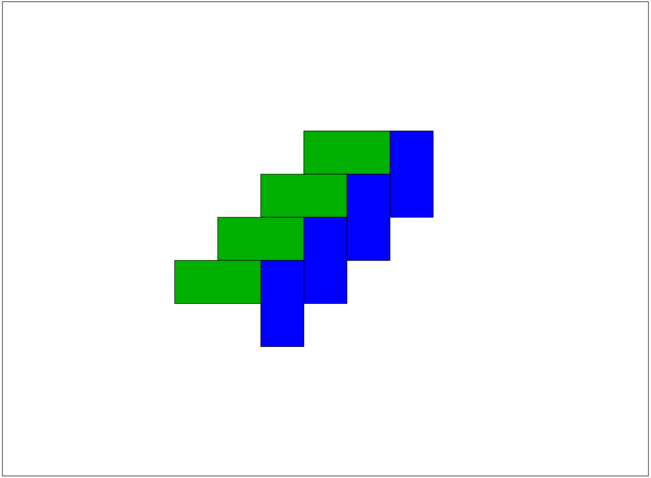
As is this.



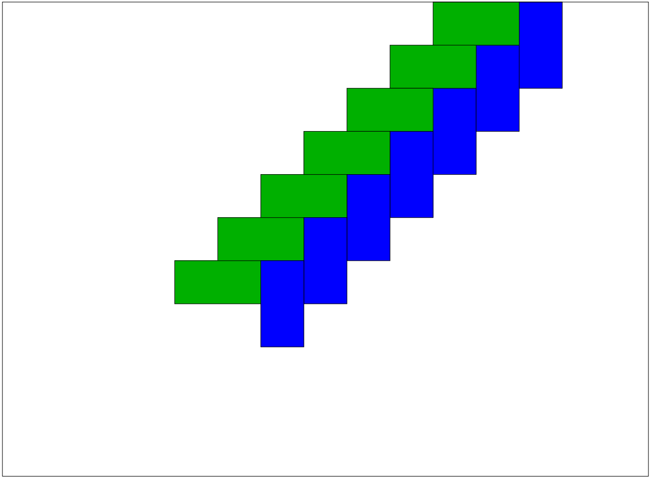
And this.



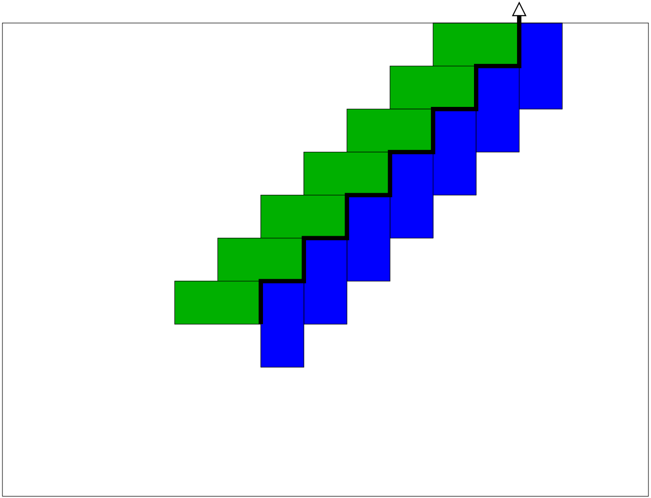
Ditto.



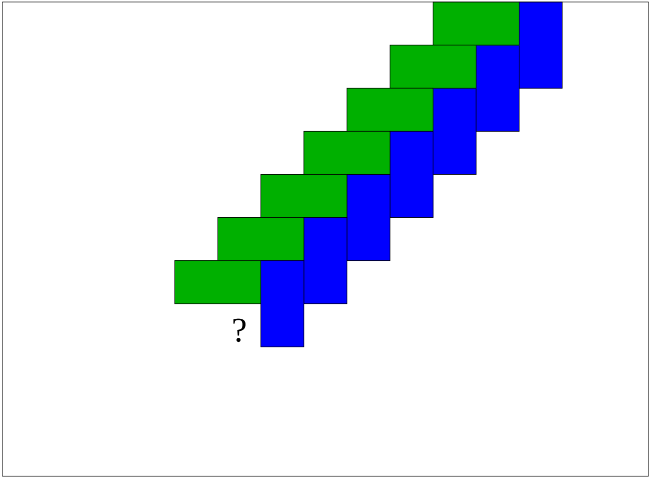
Etc.



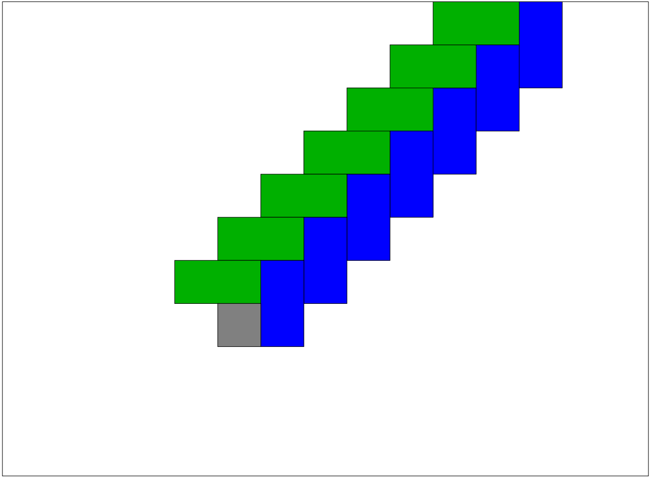
Until we reach the perimeter.



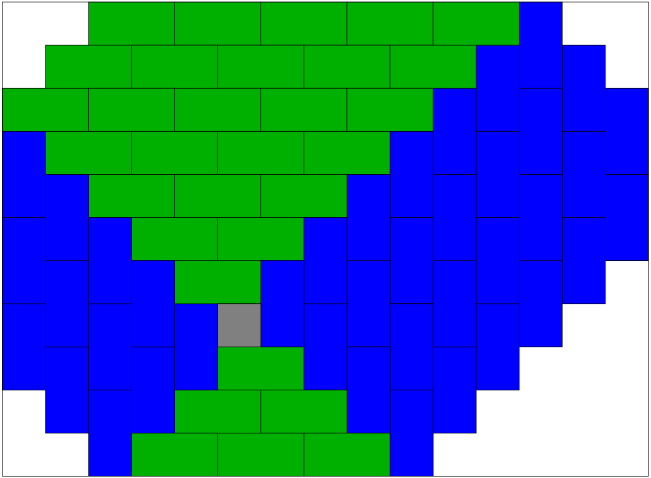
This a **ray**. They can go NE, NW, SE, SW.



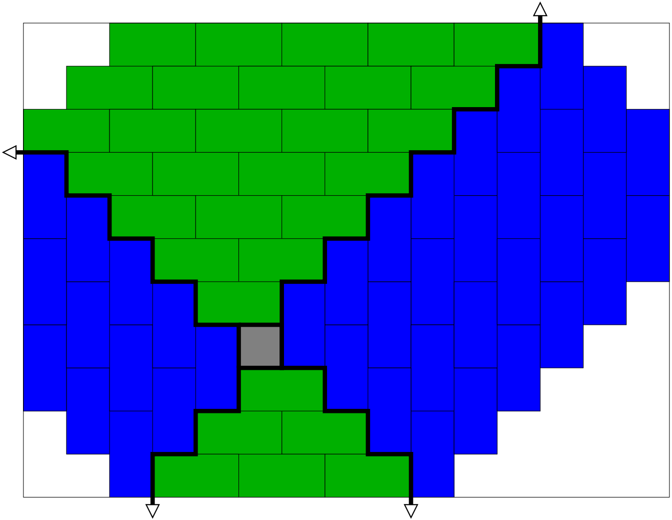
How do rays start? (The question mark.) Not a vertical dimer.



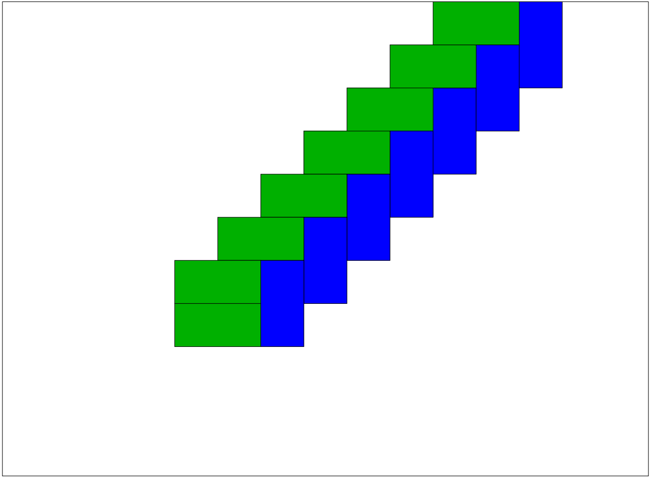
Could be a monomer.



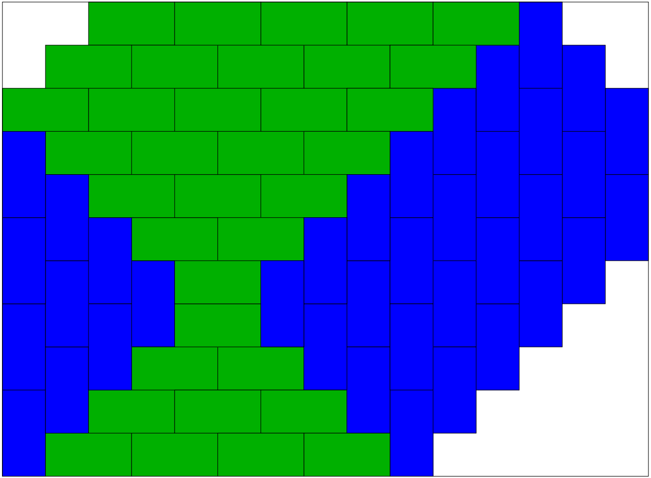
A monomer gives a **vortex**.



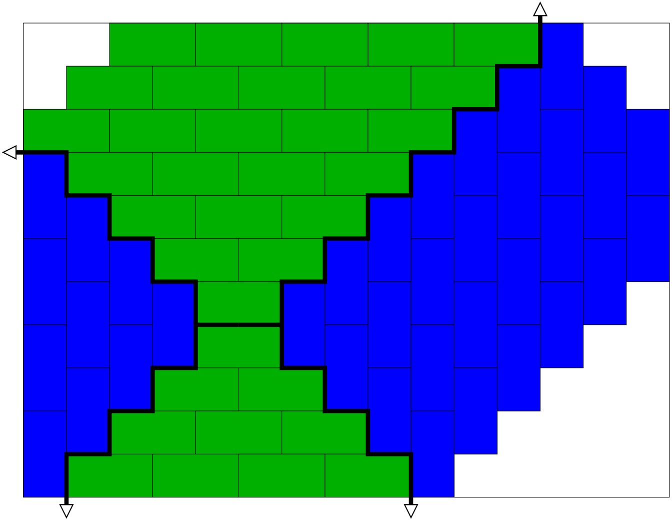
A vortex generates four rays.



Could be a horizontal dimer.



Again the placement of many tiles is forced.



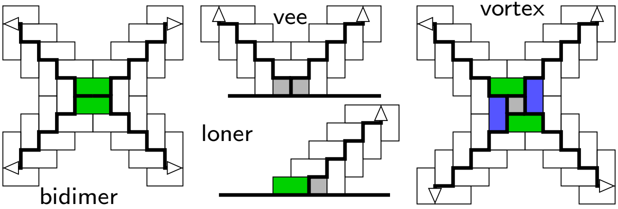
A bidimer also generates four rays.

The “beginning” of a ray: 

- Not the beginning .

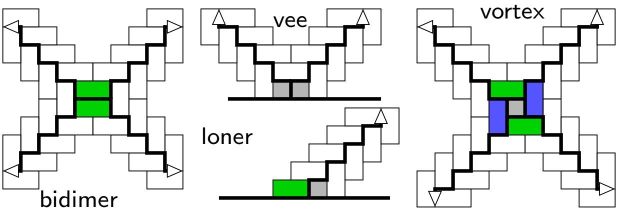
The “beginning” of a ray: 

- Not the beginning .
- Case 1, **bidimer**, two dimers share a long edge: . Occurs anywhere.

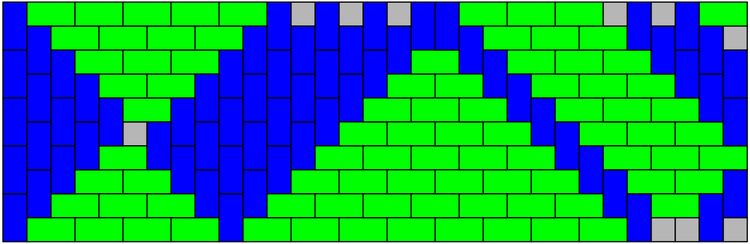


The “beginning” of a ray: 

- Not the beginning .
- Case 1, **bidimer**, two dimers share a long edge: . Occurs anywhere.
- Case 2: monomer at beginning .
  - Case 2(a), **vortex**: . Not on boundary.
  - Case 2(b), **loner**: . Only on boundary.
  - Case 2(c), **vee**: . Only on boundary.



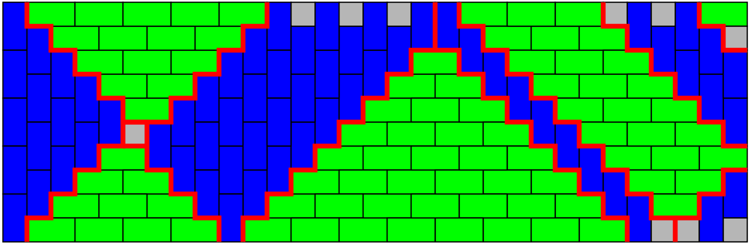
A bigger example



The border determines the diagram.

A bigger example

T-diagram



The border determines the diagram.

Tilings with no monomers.

- Let r be the number of rows and c the number of columns in an $r \times c$ rectangle.

Tilings with no monomers.

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- How many tilings, if any, are there with no monomers? Denote the number $T(r, c, 0)$.

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- The bidimers need to be carefully placed, close to the center.

Tilings with no monomers.

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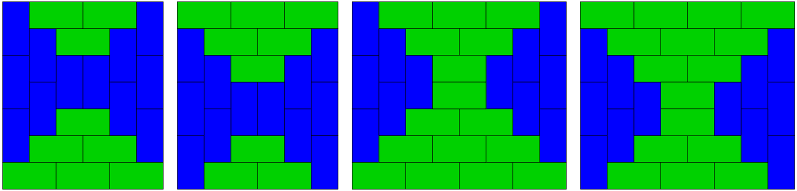
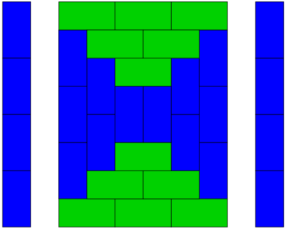
- How many tilings, if any, are there with no monomers? Denote the number $T(r, c, 0)$.
- There are no vortices (obviously).
- The bidimers need to be carefully placed, close to the center.
- If there is one bidimer only, then the dimensions must have the form $n \times n$, or $n \times (n+1)$, or $n \times (n+2)$, subject to parity constraints.

Possible tilings with one bidimer:

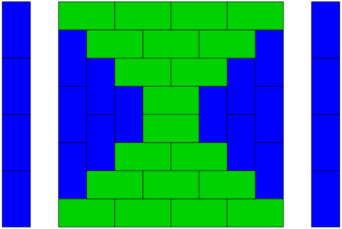
Odd height:



Even height:

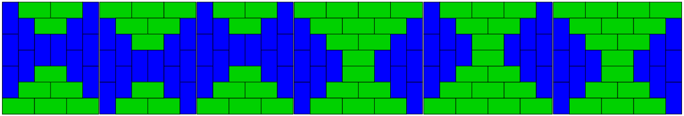


or



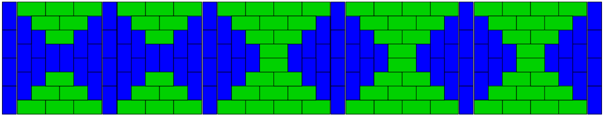
Tatami tilings and compositions (Hickerson)

Odd: Composition of c into parts of sizes $r+1$ or $r-1$. (Multiply by 2 to get tatami count.)
Example with $r = 7$:



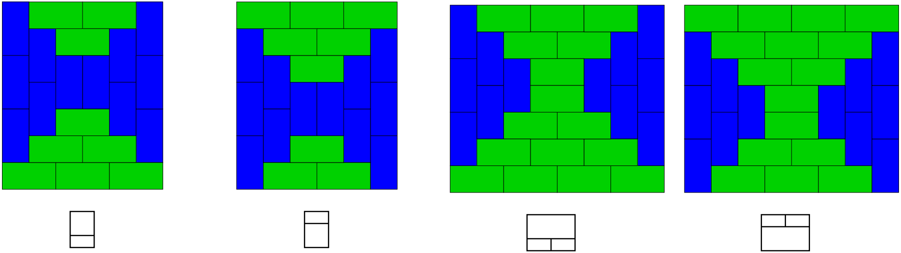
$$42 = 6 + 6 + 6 + 8 + 8 + 8$$

Even: Compositions of c into parts of sizes ($r-2$ or r) alternating with parts of size 1.
Example with $r = 8$:

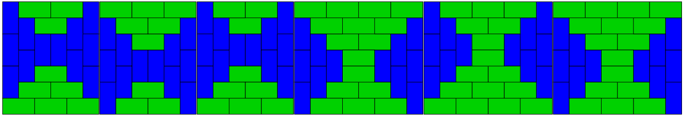


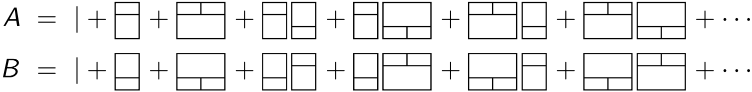
$$32 = 1 + 6 + 1 + 6 + 1 + 8 + 1 + 8 + 1 + 8 + 1$$

Encoding of odd height tilings

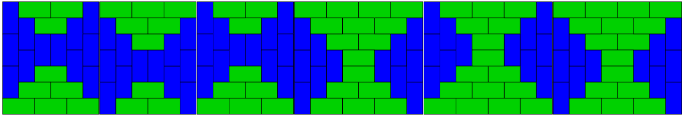


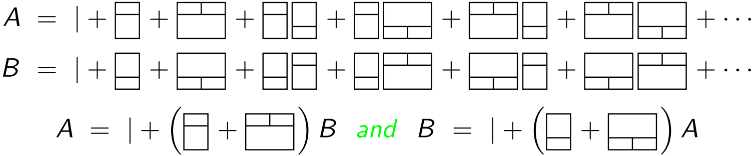
All height r tilings (for odd $r \geq 3$)



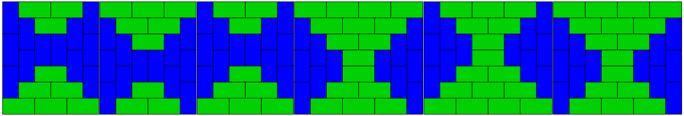


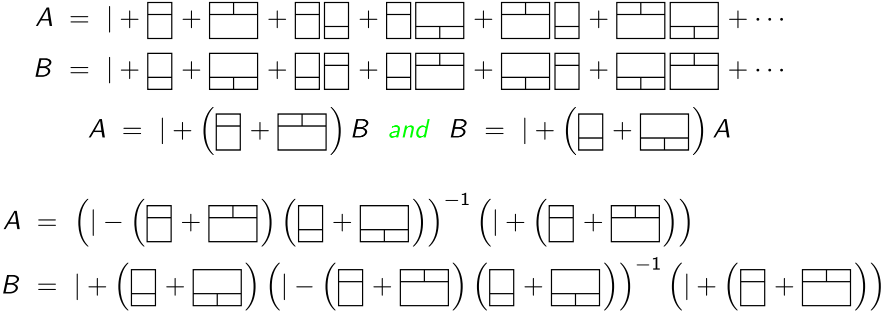
All height r tilings (for odd $r \geq 3$)



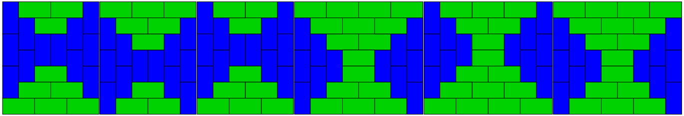


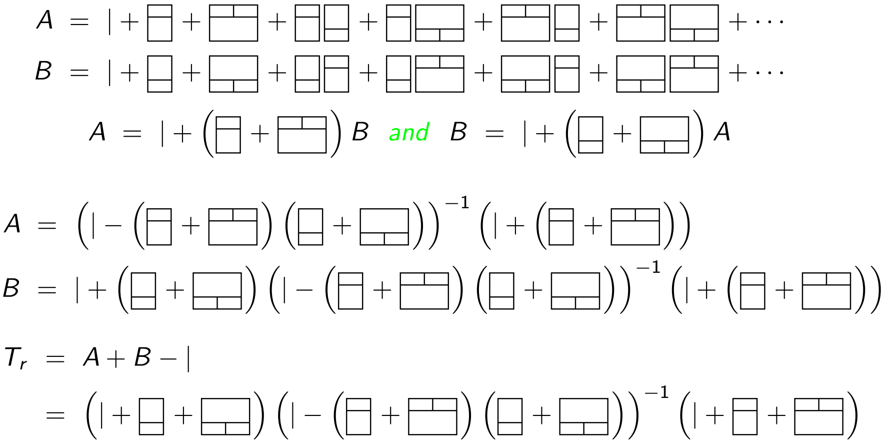
All height r tilings (for odd $r \geq 3$)



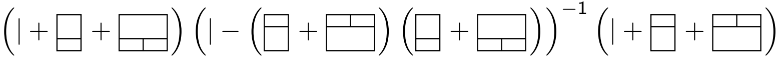


All height r tilings (for odd $r \geq 3$)

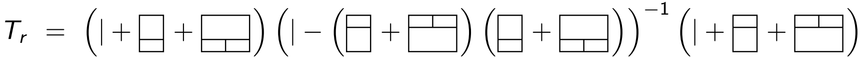




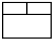
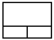
All height r tilings (odd)

$T_r =$ 

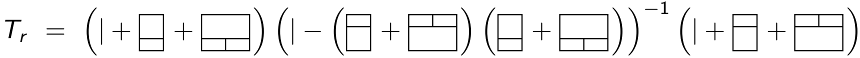
All height r tilings (odd)



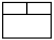
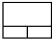
Substitute:

- 1 for $|$,
- z^{r-1} for  and , and
- z^{r+1} for  and 

All height r tilings (odd)

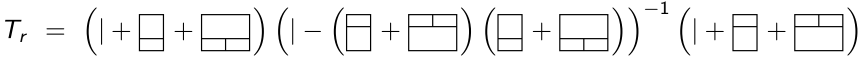


Substitute:

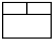
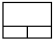
- 1 for $|$,
- z^{r-1} for  and , and
- z^{r+1} for  and 

$$T_r(z) = \left(1+z^{r-1}+z^{r+1}\right)\left(1-(z^{r-1}+z^{r+1})^2\right)^{-1}\left(1+z^{r-1}+z^{r+1}\right)$$

All height r tilings (odd)

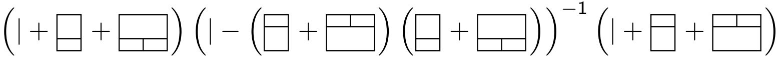


Substitute:

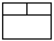
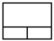
- 1 for $|$,
- z^{r-1} for  and , and
- z^{r+1} for  and 

$$
\begin{aligned}
T_r(z) &= \left(1+z^{r-1}+z^{r+1}\right)\left(1-(z^{r-1}+z^{r+1})^2\right)^{-1}\left(1+z^{r-1}+z^{r+1}\right) \\
&= \frac{(1+z^{r-1}+z^{r+1})^2}{1-(z^{r-1}+z^{r+1})^2}
\end{aligned}
$$

All height r tilings (odd)

$T_r =$ 

Substitute:

- 1 for $|$,
- z^{r-1} for  and , and
- z^{r+1} for  and 

$$\begin{aligned} T_r(z) &= \left(1+z^{r-1}+z^{r+1}\right)\left(1-(z^{r-1}+z^{r+1})^2\right)^{-1}\left(1+z^{r-1}+z^{r+1}\right) \\ &= \frac{(1+z^{r-1}+z^{r+1})^2}{1-(z^{r-1}+z^{r+1})^2} \\ &= \frac{1+z^{r-1}+z^{r+1}}{1-z^{r-1}-z^{r+1}} \end{aligned}$$

Counts by type of dimer (r odd)

With $\ell = (r+1)/2$, the coefficient of $x^h y^v$ is the number of tilings with h horizontal dimers and v vertical dimers in $T_r(x,y)$ below.

$$T_r(x,y) = \frac{1 + y^{\ell(\ell-1)} x^{(\ell-1)^2}(1 + x^{2\ell-1})}{1 - y^{\ell(\ell-1)} x^{(\ell-1)^2}(1 + x^{2\ell-1})}$$

Theorem

For r odd, the number of tatami tilings with $k(\ell^2 - \ell) = k(r^2-1)/4$ vertical and $k(\ell-1)^2 + j(2\ell-1) = k(r-1)^2/4 + jr$ horizontal tiles is

$$2\binom{k}{j}.$$

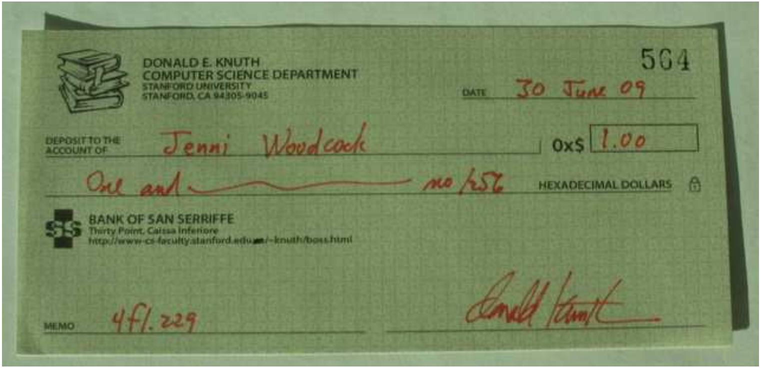
Summary: generating functions for tatami tilings of height r

$$T_r(z) = \begin{cases} 1 & \text{if } r = 0 \\ \dfrac{1}{1 - z^2} & \text{if } r = 1 \\ \dfrac{1 + z^2}{1 - z - z^3} & \text{if } r = 2 \\ \dfrac{1 + z^{r-1} + z^{r+1}}{1 - z^{r-1} - z^{r+1}} & \text{if } r \text{ odd, } 3 \le r \le c \\ (1+z)\dfrac{1+z^{r-2}+z^r}{1-z^{r-1}-z^{r+1}} & \text{if } r \text{ even, } 4 \le r \le c \end{cases}$$

Consequences (errata and check)

Page 229 last lines of answer 215 ——— 10 Jun 2009

413–420.] . . . $(1 - z^{m-1} - z^{m+1})$. $\leadsto$ 413–420. The set of all tatami tilings has been characterized by Dean Hickerson; the corresponding generating functions have been obtained by Frank Ruskey and Jennifer Woodcock, *Electronic J. Combinatorics* **16**, 1 (2009), #R126.]



Changing gears!

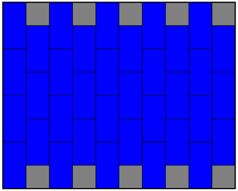
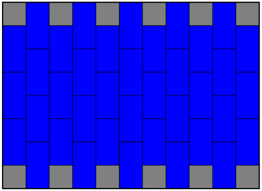
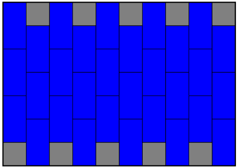
We were looking at tatami tilings where $m = 0$ (no monomers).

Now we look at tatami tilings where m is maximized.

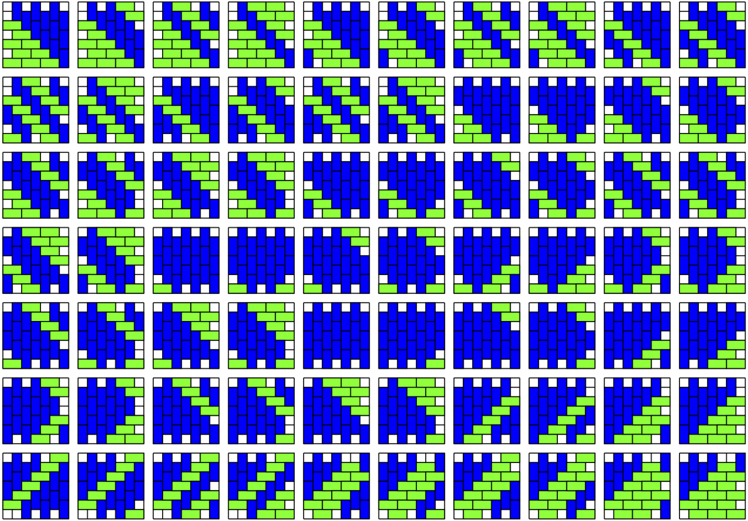
What is the maximum number of monomers?

Assuming $r \leq c$,

$$\max = \begin{cases} c+1 & \text{if } r \text{ even }, c \text{ odd} \\ c & \text{otherwise.} \end{cases}$$

Some 7×7 tilings with 7 monomers



Maximizing monomers in a square

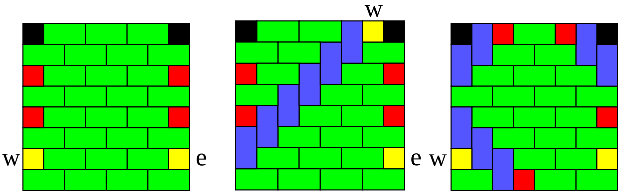
Theorem: There are $n2^{n-1}$ tatami tilings of an $n \times n$ square that use the maximum number of monomers.

Proof: First, structural preliminaries.

- An $n \times n$ tatami tiling has n monomers () if and only if it has no vortices () or bidimers ().
- An $n \times n$ tatami tiling can not have more than n monomers.
- The trivial tiling has n monomers.
- Every other tiling with n monomers can be obtained from the trivial tiling by *diagonal flips*.
- Every tiling with n monomers has monomers in two adjacent corners.

Counting part of proof (n even)

Upper corners fixed,
a diagonal flip,
3 diagonal flips.



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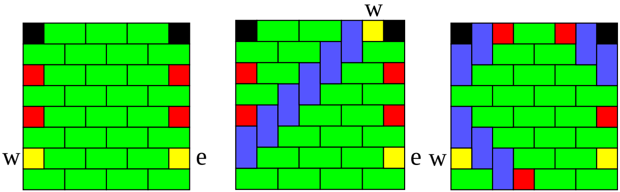
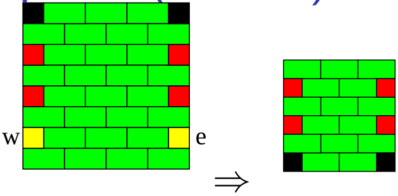


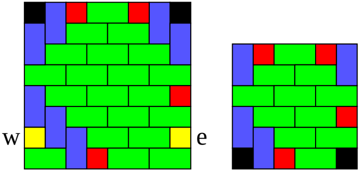
- Classification: w, e flip up or not?
- If yes, then all flips have the same orientation.
- $n-3$ other flips possible.
- Contribution to the count $2 \cdot 2^{n-3}$.

Counting part of proof (n even)

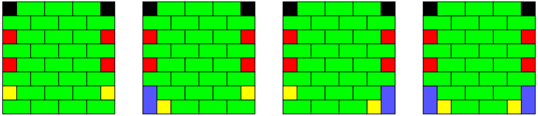
If no, then *reduce*:



Non-trivial example:

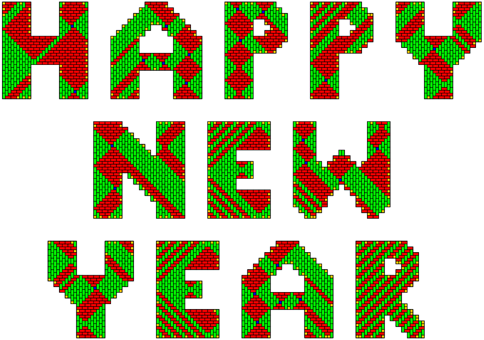


w, *e* can flip down:



Recurrence relation: $s(n) = 2 \cdot 2^{n-3} + 4 \cdot s(n-2)$.

The happy new year message again



Dear Friends,
... If you are bored, perhaps you will have fun proving that the number of tilings of an $n \times n$ square that maximize the number of monomers is $n2^{n-1}$.
Cheers, Frank

Don Knuth e-mail

Dear Frank,
I resisted the challenge in your New Year's card (about $2^{n-1}n$) for more than four weeks, but finally realized that I couldn't live any longer without trying to find out what was going on with those tatami tilings. I budgeted half a day to explore the problem; and finally figured out enough of the structure to declare victory after two days; but my derivation is not at all simple. Certainly I have no way to group the solutions into, say, n classes of size 2^{n-1} (although I do have lots of classes of solutions of size 2^{n-2}).
....
All lots of fun, but I do have to get back to TAOCP!
Cordially, Don

Previously open problem: Give a direct bijective proof that has "n classes of size 2^{n-1}" (or vice-versa). Solved by Mark Schurch.

Extension to rectangles ($r \leq c$)

Theorem

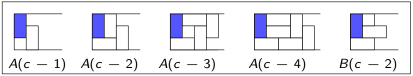
$T(r, c, \max)$ *is equal to*

$$
\begin{cases}
2^{r-4}(r+4)(r+2) & \text{if } r \text{ even}, c \text{ even}, c > 2r+1 \\
2^{r-4}(r+3)^2 & \text{if } r \text{ odd}, c > 2r+1 \\
2^{r-6}(r+2)^2 & \text{if } r \text{ even}, c \text{ odd}, c > 2r+1 \\
2^{r-4}\,(3r{-}c{+}4)\,(c{-}r{+}2) & \text{if } r \equiv c \bmod 2 \text{ and } r+1 \leq c \leq 2r+1 \\
2^{r-4}\,(3r{-}c{+}4)\,(c{-}r{+}2) - 2^{r-4} & \text{if } r \text{ odd}, c \text{ even}, r+2 \leq c \leq 2r \\
2^{r-6}\,(29r{+}17) & \text{if } r \text{ odd}, c \text{ even}, c = r+1 \\
2^{r-6}\,(3r{-}c{+}4)\,(c{-}r{+}2) & \\
\quad +(2c{-}2r{+}3)2^{r-6} & \text{if } r \text{ even } c \text{ odd}, r{+}1 \leq c \leq 2r{+}1 \\
r2^{r-1} & \text{if } r = c
\end{cases}
$$

Monomers arbitrary, but r fixed

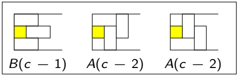
For height $r = 3$

Let $A(c)$ be the number of $3 \times c$ tilings which start with the blue tile shown on the right.

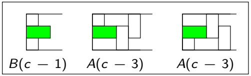


Recurrences for $A(c)$

Similarly for $B(c)$ and $C(c)$



Recurrences for $B(c)$

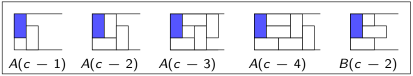


Recurrences for $C(c)$

Monomers arbitrary, but r fixed

For height $r = 3$

Let $A(c)$ be the number of $3 \times c$ tilings which start with the blue tile shown on the right.



Recurrences for $A(c)$

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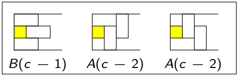
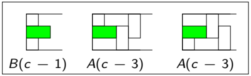


Recurrences for $B(c)$



Recurrences for $C(c)$

$$A(c) = A(c-1) + A(c-2) + A(c-3) + A(c-4) + B(c-2),$$
$$B(c) = B(c-2) + 2A(c-2),$$
$$C(c) = B(c-1) + 2A(c-3).$$

This is a linear recurrence relation in A, B, C so we have rational generating functions.
The number of tilings with r rows and c columns is the coefficient of z^c in the generating function $T_r(z)$.

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Theorem

For height $r = 1, 2, 3$ the generating functions $T_r(z)$ are

$$T_1(z) = \frac{1}{1 - z - z^2}, \qquad T_2(z) = \frac{1 + 2z^2 - z^3}{1 - 2z - 2z^3 + z^4}, \quad \textit{and}$$

$$T_3(z) = \frac{1 + 2z + 8z^2 + 3z^3 - 6z^4 - 3z^5 - 4z^6 + 2z^7 + z^8}{1 - z - 2z^2 - 2z^4 + z^5 + z^6}.$$

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Note that the denominators are (almost) self-reciprocal. **Open:** Why is this true?

The g.f. for $n = 10$, from Maple

$$T_{10}(z) = (224z^{65} + 280z^{64} - 54z^{63} - 768z^{62} - \cdots \text{ (terms omitted) } \cdots - 3270z^{10} + 1239z^{9} - 3570z^{8} + 1814z^{7} - 2824z^{6} + 815z^{5} - 4676z^{4} - 678z^{3} + 4240z^{2} + 88z + 1) \;\; / \;\; (z^{56} - z^{55} - z^{54} + z^{53} - z^{52} + z^{51} - z^{50} + z^{49} - z^{48} - 4z^{47} + 4z^{46} - 16z^{45} + 15z^{44} + z^{43} - z^{42} + 17z^{41} - 17z^{40} + 33z^{39} - 23z^{38} + 41z^{37} + 7z^{36} - 2z^{35} + 66z^{34} - 66z^{33} + 18z^{32} - 18z^{31} - 78z^{30} + 68z^{29} - 120z^{28} + 68z^{27} - 78z^{26} - 18z^{25} + 18z^{24} - 66z^{23} + 66z^{22} - 2z^{21} + 7z^{20} + 41z^{19} - 23z^{18} + 33z^{17} - 17z^{16} + 17z^{15} - z^{14} + z^{13} + 15z^{12} - 16z^{11} + 4z^{10} - 4z^{9} - z^{8} + z^{7} - z^{6} + z^{5} - z^{4} + z^{3} - z^{2} - z + 1)$$

A table of the numbers

r \ c	1	2	3	4	5	6	7	8	9	10
1	1	2	3	5	8	13	21	34	55	89
2	2	6	13	29	68	156	357	821	1886	4330
3	3	13	22	44	90	196	406	852	1778	3740
4	5	29	44	66	126	238	490	922	1714	3306
5	8	68	90	126	178	325	584	1165	2030	3619
6	13	156	196	238	325	450	827	1404	2828	4603
7	21	357	406	490	584	827	1090	1914	3262	6228
8	34	821	852	922	1165	1404	1914	2562	4618	7450
9	55	1886	1778	1714	2030	2828	3262	4618	5890	10130
10	89	4330	3740	3306	3619	4603	6228	7450	10130	13314

- Note that the numbers are non-monotone.
- How fast are they growing?

Asymptotics

The roots of the denominator of $T_2(z)$ are

$$\frac{1}{2}\left(1-\sqrt{3}\pm\sqrt{-2\sqrt{3}}\right) \text{ and } \frac{1}{2}\left(1+\sqrt{3}\pm\sqrt{+2\sqrt{3}}\right).$$

The one with the smallest modulus is

$$\frac{1}{\beta}=\frac{1}{2}\left(1+\sqrt{3}-\sqrt{2\sqrt{3}}\right)\approx 0.435420544682339\ldots$$

Asymptotically,

$$T(2,n)\sim\frac{-\beta P(1/\beta)}{Q'(1/\beta)}\beta^n = 1.0607\ldots(2.2966\ldots)^n$$

The corresponding value of $1/\beta$ for $T_3(z)$ is $2.0953\cdots$.

r	p_r Num.	q_r Den.	Coefficients of the Denominator (ascending degree)
1	1	2	$1, -1, 1$
2	3	4	$1, -2,$ **0**$, -2, 1$
3	8	6	$1, -1, -2,$ **0**$, -2, 1, 1$
4	14	11	$-1, 1, 1, 1, -1,$ **7,-7**$, 1, -1, -1, -1, 1$
5	18	14	$-1, 1, 1, -1, 3, -1, 5,$ **-2**$, -5, -1, -3, -1, -1, 1, 1$
6	27	22	$1, -1, -1, 1, -1, -2, 2, -10, 9, -1, 4,$ **6**$,$ $4, -1, 9, -10, 2, -2, -1, 1, -1, -1, 1$
7	28	22	$1, -1, -3, 3, 4, -4, -9, 7, 6, -5, 2,$ **0**$,$ $2, 5, 6, -7, -9, 4, 4, -3, -3, 1, 1$
8	44	37	$-1, 1, 1, -1, 1, -1, 1, 3, -3, 13, -12, 0, 0, -12,$ $6, -20, -6, 2,$ **-34,34**$, -2, 6, 20, -6, 12, 0, 0, 12,$ $-13, 3, -3, -1, 1, -1, 1, -1, -1, 1$
9	50	42	$-1, 1, 1, -1, 1, -1, 1, -1, 5, -3, 11, -8, -6, 4,$ $-14, 8, -20, 2, -28, 2, -24,$ **10**$, 24, 2, 28, 2, 20, 8$ $14, 4, 6, -8, -11, -3, -5, -1, -1, -1, -1, -1, -1, 1, 1$
10	65	56	$1, -1, -1, 1, -1, 1, -1, 1, -1, -4, 4, -16, 15, 1, -1,$ $17, -17, 33, -23, 41, 7, -2, 66, -66, 18, -18, -78, 68,$ **-120**$,$ $68, -78, -18, 18, -66, 66, -2, 7, 41, -23, 33, -17,$ $17, -1, 1, 15, -16, 4, -4, -1, 1, -1, 1, -1, 1, -1, -1, 1,$

Conjectures: Let $T_r(z) := P_r(z)/Q_r(z)$, where $P_r(z)$ and $Q_r(z)$ are relatively prime polynomials, and $q_r := \deg(Q_r(z))$.

$$Q_r(z) = \begin{cases} -z^{q_r} Q_r(+1/z) & \text{if } r \equiv 0(4) \\ -z^{q_r} Q_r(-1/z) & \text{if } r \equiv 1(4) \\ +z^{q_r} Q_r(+1/z) & \text{if } r \equiv 2(4) \\ +z^{q_r} Q_r(-1/z) & \text{if } r \equiv 3(4) \end{cases}$$

Let $s = \lfloor r/2 \rfloor$.

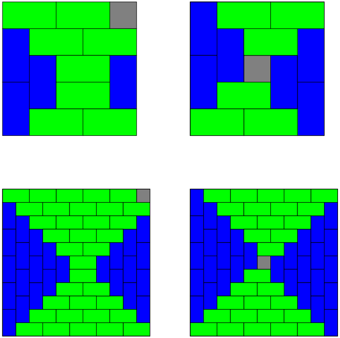
$$q_r = \begin{cases} 1 + s + 2s^2 & \text{if } r \equiv 0, 2, 3(4) \\ 2 + 2s + 2s^2 & \text{if } r \equiv 1(4) \end{cases}$$

Open Problem

Conjecture: For all $k \geq 0$ and $m \geq 1$ there is an n_0 such that, for all $n \geq n_0$ where $n(n+k)$ and m have the same parity,

$$T(n+k, n, m) = T(n_0+k, n_0, m).$$

Example ($k = 0, m = 1$):
$T(n, n, 1) = 8 + 2 = 10$ for all $n \geq 3$.



Formula for m monomers in a square

- **The $k = 0$ case (is true):**
The number of $n \times n$ tatami tilings with m monomers, $m = n \bmod 2$, and $m < n$, is

$$T(n, n, m) = (m + 1)2^{m+1} + m2^m.$$

- **Example:** When $m = 1$, we get $2 \cdot 2^2 + 2 = 10$ (previous slide).

Counting by type of dimer

- Let $K_n(z)$ be the polynomial whose i-th coefficient is the number of $n \times n$ tilings with n monomers and that contain i vertical dimers.
- **For example:**

$$\begin{aligned} K_{11}(z) &= 2(1+z)^5(1+z^2)^2(1-z+z^2)(1+z^4)(1-z+z^2-z^3+z^4)I(z) \\ &= \Phi_2^5(z)\Phi_4^2(z)\Phi_6(z)\Phi_8(z)\Phi_{10}(z)I(z) \end{aligned}$$

  where $I(z)$ is irreducible and $\Phi_d(z)$ is the d-cyclotomic polynomial.
- Knuth: "... so something is indeed going on, cyclotomically!"
- Conjecture: For n even,

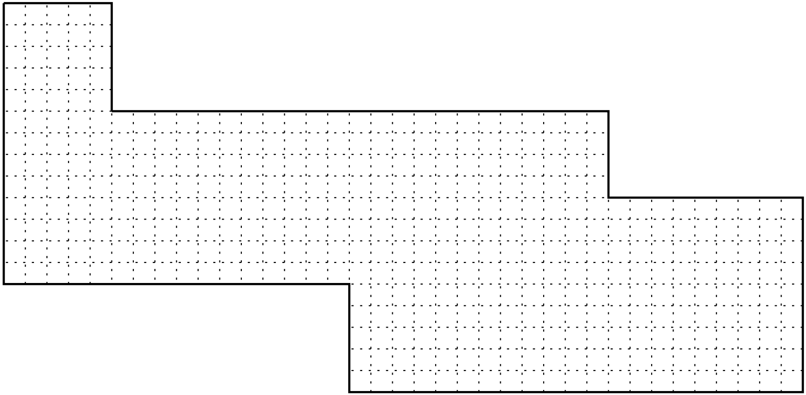
$$K_n(z) = I_n(z) \prod_{j \geq 1} S_{\lfloor (n-1)/2^j \rfloor}(z),$$

  where $I_n(z)$ is an irreducible polynomial and

$$S_n(z) := (1+z)\cdots(1+z^n) = \prod_{j=1}^{n} \Phi_{2j}^{\lfloor \frac{n+j}{2j} \rfloor}(z).$$

More tatami problems

Given an arbitrary shaped grid, what is the minimum number of monomers in a tatami tiling?



Is there a polynomial-time algorithm to determine the answer?

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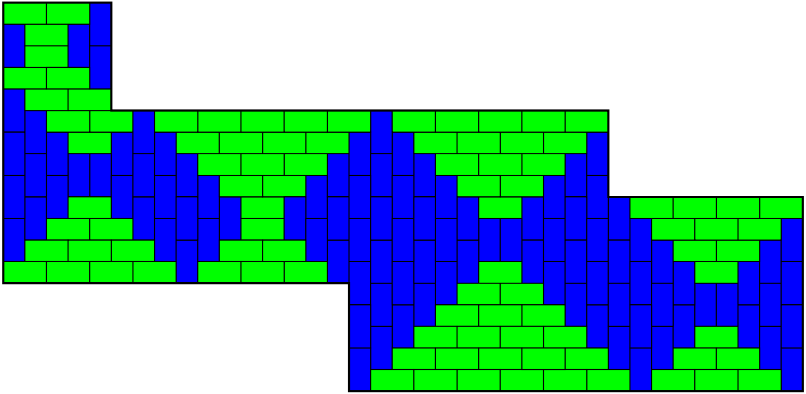
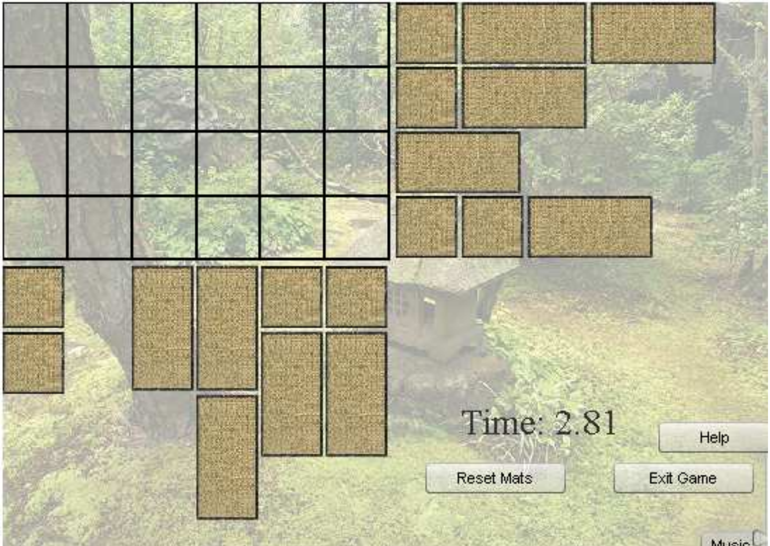


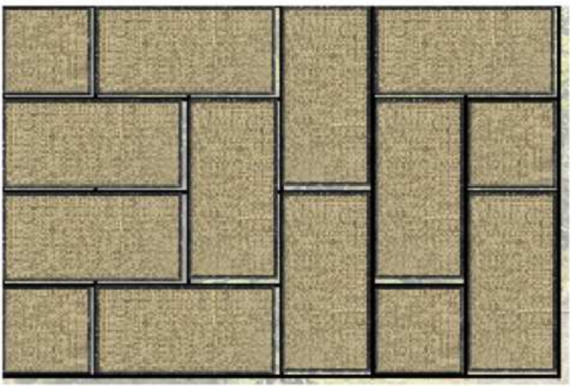
Is there a polynomial-time algorithm to determine the answer?

Tomography

Is it possible to tile a grid with these row and column projections? What is the complexity of this?

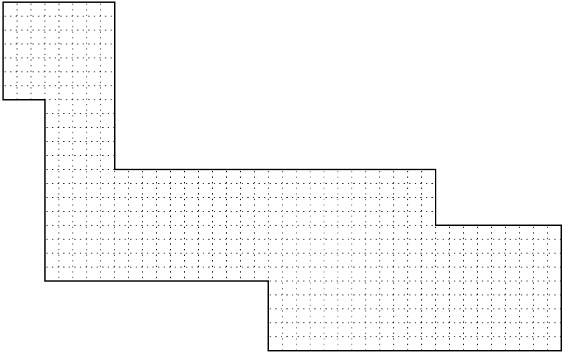


Alejandro's flash game



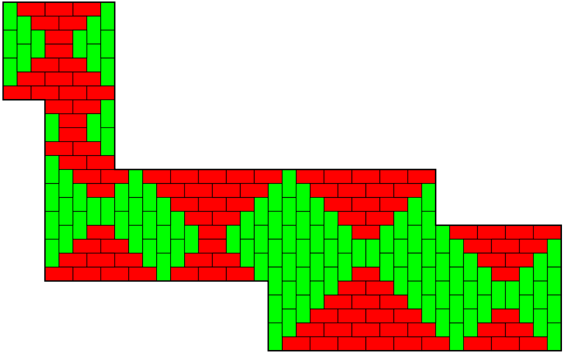
Play this flash game at
`http://miniurl.org/tomoku`.

Magnetic water strider problem



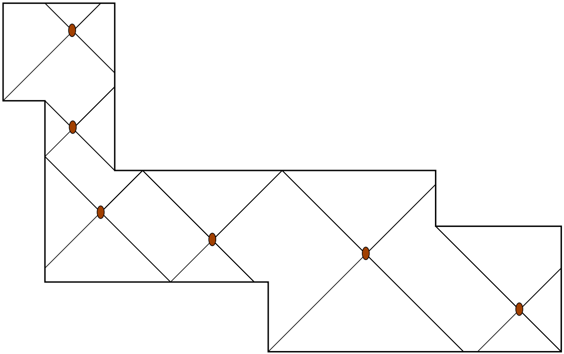
- Strider legs can not cross.

Magnetic water strider problem



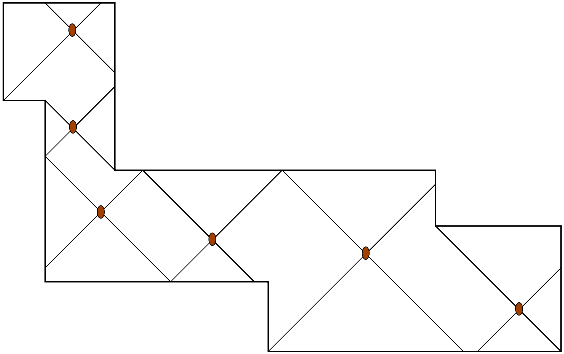
- Strider legs can not cross.

Magnetic water strider problem



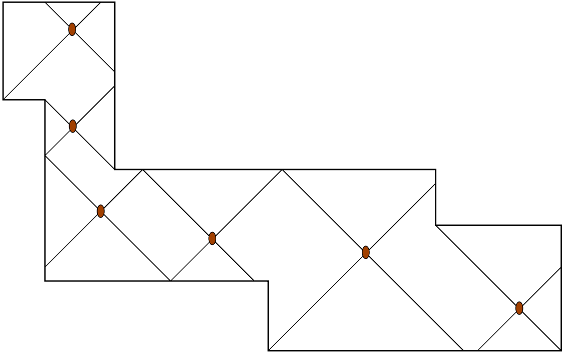
- Strider legs can not cross.
- How many striders can be placed?

Magnetic water strider problem



- Strider legs can not cross.
- How many striders can be placed?
- Find a minimum strider maximal configuration.

Magnetic water strider problem



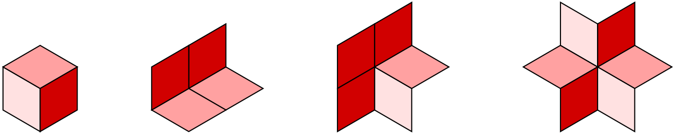
- Strider legs can not cross.
- How many striders can be placed?
- Find a minimum strider maximal configuration.
- Game: players take turns placing striders.

Lozenge Tatami Tilings?

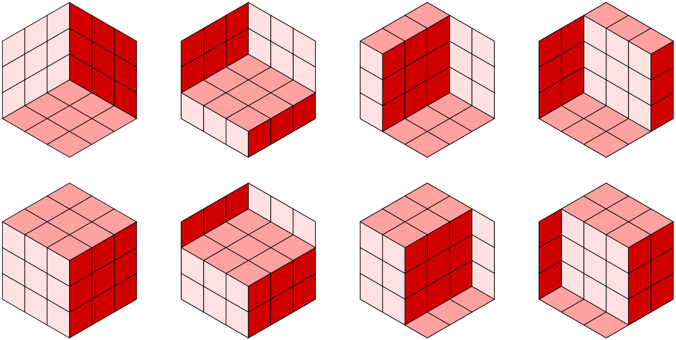
- This work done together with Jen Debroni.
- We consider lozenge tilings of n by m by k hexagons.
- With no constraints, the number of tilings has a beautiful formula (MacMahon):

$$L_6(k, n, m) = \prod_{i=1}^{k} \prod_{j=1}^{n} \prod_{\ell=1}^{m} \frac{i + j + \ell - 1}{i + j + \ell - 2}$$

- The number of lozenges that can meet at a grid point is 3, 4, 5, or 6.



- $L_3(k, n, m)$ is 0 unless $k = n = m = 1$.
- $L_4(k, n, m) = k + n + m - 1$.



- The interesting case is $L_5(k, n, m)$.
- $L(1, 1, n) = n + 1$.
- $L(2, 2, n) = \frac{1}{12}(1 + n)(12 + 18n + 5n^2 + n^3)$.
- Conjecture: For fixed k and m the value $L(k, m, n)$ is a polynomial in n of degree km.

Thanks for coming!
Any questions?